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ROOT SELECTIONS AND 2^p-th ROOT SELECTIONS IN HYPERFIELDS

Paweł Gładki

Institute of Mathematics
University of Silesia
ul. Bankowa 14, 40–007 Katowice, Poland

e-mail: pawel.gladki@us.edu.pl

Abstract

In this paper we define root selections and 2^p-th root selections for hyperfields: these are multiplicative subgroups whose existence is equivalent to the existence of a well behaved square root function and 2^p-th root function, respectively. We proceed to investigate some basic properties of such root selections, and draw some parallels between the theory of root selections for hyperfields and the theory of orderings and orderings of higher level in hyperfields previously studied by the author.

Keywords: square roots, 2^p-th roots, orderings, orderings of higher level, hyperfields.

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1. Introduction

Preorderings and orderings in fields constitute a part of the standard algebraic curriculum that can be found in most modern textbooks. Historically they were given today's shape in a series of seminal papers [1] and [2] by Artin and Schreier where a complete solution to the celebrated Hilbert 17th problem was settled: for a field F a *preordering* is a subset $T \subseteq F$ closed under addition and multiplication that contains the set F^2 of all squares of F, that is $T + T \subseteq T$, $T \cdot T \subseteq T$, $F^2 \subseteq T$, whereas an *ordering* is a subset $P \subseteq F$ such that $P + P \subseteq P$, $P \cdot P \subseteq P$, $P \cap -P = \{0\}$ and $P \cup -P = F$, where $-P = \{-a \in F \mid a \in P\}$. Clearly every ordering is a preordering. Over the years, the notions of preorderings and orderings have been generalized in a plentitude of directions. One of such generalizations is the theory of preorderings and orderings of higher level, whose approachable exposition can be found in classical monographs [3] by Becker and [10] by Lam: a *preordering of level* p is a subset $T \subseteq F$ such that $T + T \subseteq T$, $T \cdot T \subseteq T$ and $F^{2^p} \subseteq T$, F^{2^p} denoting the set of 2^p-th powers in F, while an *ordering of level* p is a subset $P \subseteq F$ such that $P + P \subseteq P$, P^* is a subgroup of the multiplicative group F^* of F (for a subset $A \subseteq F$, A^* shall always denote the set $A \setminus \{0\}$), and F^*/P^* is a cyclic group of order 2^m with $m \leq p$ – if $m = p$, we say P is of *exact level* p. Again, an ordering of level p is always a preordering of level p.

As seen already from the above definitions, squares and, respectively, 2^p-th powers play a central role in (pre-) orderings and (pre-) orderings of higher level. Taking into consideration the multiplicative group F^{*2} of squares of F, it is somewhat natural to ask when it is possible to define a square root function that behaves reasonably well, that is which is a homomorphism $\phi : F^{*2} \to F^*$ which maps a square c^2 to c multiplied by a "sign", that is such that $\phi(c^2) = \omega c$, where $\omega^2 = 1$. This question was first addressed by Waterhouse in [12] and it turns out that the existence of such a homomorphism is closely related to the existence of orderings. Firstly, a homomorphism $\phi : F^{*2} \to F^*$ such that $\phi(c^2) = \omega c$, where $\omega^2 = 1$, exists if and only if there is a subgroup R of F^* called *root selection* such that every element of F^* can be uniquely represented as ωr with $\omega^2 = 1$ and $r \in R$ ([12], Lemma, p. 235). Secondly, a root selection exists if and only if -1 is not a square in F ([12], Theorem 1); since, by classical theorems due to Artin and Schreier [1, 2], an ordering in F exists if and only if -1 is not a sum of squares in F, it follows that root selections exist in every ordered field (but, of course, also outside of them, the simplest example being $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$, so that $-1 = q - 1 \notin \mathbb{F}_q^{*2}$), and thus can be perceived as generalizations of orderings. Hence a small but neat theory of fields with root selections can be built somewhat parallel to the theory of ordered fields, where issues such as existence of root selections (that we have just briefly outlined), extensions of fields with root selections, and structure of maximal root selection fields (somewhat corresponding to real closed fields) are discussed – all of this was essentially done by Waterhouse in [12].

Most of the results of [12] generalize in an elegant way to the multiplicative group F^{*2^p} of 2^p-th powers of F and lead to the consideration of the existence of a reasonably well behaved 2^p-th root function. In a miniature note [4] by the author it has been shown that, for a field F containing the 2^p-th primitive root of unity ω_{2^p}, a homomorphism $\phi : F^{*2^p} \to F^*$ such that $\phi(c^{2^p}) = \omega_{2^p}^k c$, for some $k \in \{1, \ldots, 2^p\}$ exists if and only if there is a multiplicative subgroup R of F^*, called 2^p*-th root selection*, such that every element of F^* can be uniquely represented as $\omega_{2^p}^k r$ with $k \in \{1, \ldots, 2^p\}$ and $r \in R$ ([4], Lemma 2.1), and that 2^p-th root selections exist if and only if -1 is not a 2^p-th power in F ([4], Theorem 2.4). Therefore a tiny theory parallel to the one of orderings of higher level can be built, and, in particular, questions relevant to the existence of 2^p-th root selections, or to extensions of fields with 2^p-th root selections, or to the structure of maximal 2^p-th root selection fields can be addressed.

On the other hand, the notion of orderings (and orderings of higher level) can be carried to other than fields algebraic structures. Out of a plethora of possibilities, we shall focus on the particular case of algebras with multivalued addition that resemble fields and are thus called *hyperfields*. It is hard to say who first considered such objects, as their definition is very natural, but most sources point to Krasner as one of the founding fathers of the theory of hyperfields and his work on valuations [9]. By a hyperfield we shall understand a system $(H, +, \cdot, -, 0, 1)$, where $+ : H \times H \to 2^H$ is the multivalued addition, $\cdot : H \times H \to H$ is the usual multiplication, $- : H \to H$ is the subtraction function, and $0, 1 \in H$ are elements such that the following axioms hold:

(i) $\forall a, b, c \in H[a + (b + c) = (a + b) + c]$,

(ii) $\forall a \in H[a + 0 = 0 + a = \{a\}]$,

(iii) $\forall a, b, c \in H[a \in b + c \Longrightarrow b \in a + (-c)]$,

(iv) $\forall a, b \in H[a + b = b + a]$,

(v) $(H^*, \cdot, 1)$ is a commutative group,

(vi) $\forall a, b, c \in H[a \cdot (b + c) = a \cdot b + a \cdot c]$,

(vii) $\forall a \in H[a \cdot 0 = 0]$.

Note that, for $a, b \in H$, $a + b$ is always a set, so, in particular, $a + (b + c)$ is, in fact, equal to $\bigcup\{a + x \mid x \in b + c\}$. 0 is the neutral element of $+$. It is not possible to define inverse elements with respect to addition in a classical way, which is the reason why, instead, one introduces the subtraction function, so that a cancellation property captured by the axiom (iii) is satisfied.

Some classical properties of addition and multiplication that can be easily deduced from axioms of fields can not be proven in a similar way for hyperfields, and hence have to be added as separate axioms – in particular the axiom (vii) is not a consequence of axioms (i)–(vi). Likewise, characteristics is defined in a tricky way: $\mathrm{char}(H) = k$ if k is the least integer such that $0 \in \underbrace{1 + \cdots + 1}_{k}$, or 0 if no such k exists. Note that $\mathrm{char}(H) \neq 2$ implies $1 \neq -1$. Throughout the paper we shall only consider hyperfields of characteristics different from 2. Since, as we will see, the theory developed in the paper is closely related to the theory of formally real hyperfields, which are neccessarily of characteristics 0, this additional assumption is not really restrictive.

Hyperfields find numerous applications in the theory of quadratic forms, as they provide a convenient and natural language to axiomatize the behaviour of quadratic forms over fields: for a field F with char$F \neq 2$, $F \neq \mathbb{F}_3, \mathbb{F}_5$, consider the group of square classes F^*/F^{*2} with the usual multiplication to which we adjoin the element 0 and, for $a, b \in F^*/F^{*2}$, define multivalued addition (that extends to 0 in an obvious way) by $a + b = D(a, b)$, where $D(a, b) = \{as^2 + bt^2 \mid s, t \in F^*\}$ is the value set of the quadratic form $\langle a, b\rangle = aX^2 + bY^2$. As a result we obtain a hyperfield that shall be denoted by $Q(F)$ and called the *quadratic hyperfield* of F (this construction can be slightly ammended to include also the case of fields F with char$(F) = 2$ or with $F = \mathbb{F}_3, \mathbb{F}_5$ – see Proposition 2.1 in [7]). It is, therefore, important to define preorderings and orderings on hyperfields and ask what part of the standard Artin-Schreier theory can be carried to the multivalued case. This was essentially done by Marshall in [11]: just like in the field case, for a hyperfield H a *preordering* is a subset $T \subseteq H$ such that $T + T \subseteq T$, $T \cdot T \subseteq T$, $H^2 \subseteq T$, and an *ordering* is a subset $P \subseteq H$ such that $P + P \subseteq P$, $P \cdot P \subseteq P$, $P \cap -P = \{0\}$ and $P \cup -P = H$ (note, however, that here $P + P$ means $\bigcup\{a + b \mid a, b \in P\}$). Subsequently, preorderings and orderings of higher level in hyperfields were defined and investigated by the author in [5] and by Marshall and the author in [6]: again, for a hyperfield H, a *preordering of level* p is a subset $T \subseteq H$ such that $T + T \subseteq T$, $T \cdot T \subseteq T$ and $H^{2^p} \subseteq T$, while an *ordering of level* p is a subset $P \subseteq H$ such that $P + P \subseteq P$, P^* is a subgroup of the multiplicative group H^* of H, and H^*/P^* is a cyclic group of order 2^m with $m \leq p$ – if $m = p$, we say P is of *exact level* p. Again, an ordering (of level p) in a hyperfield is always a preordering (of level p).

In this paper we add one more piece to the above described puzzle and define root selections and 2^p-th root selections over hyperfields and continue to investigate what parts of the theory for fields can be carried to the hyperfield case. In order to make our presentation more compact, we state all of our results already for 2^p-th root selections, and obtain respective definitions and theorems for "ordinary" root selections as special cases. Basic definitions of hyperfields, their preorderings and orderings, as well as preorderings and orderings of higher level, that are required to proceed with the exposition, have already been mentioned in this Introduction, but we kindly refer the reader to [11] or to [5] and [6] as far as general background on hyperfields and their orderings is concerned. Nevertheless, whenever a definition or a nontrivial argument pertinent to the realm of multivalued algebra is needed in the course of the paper, we state it with a proper reference.

2. Existence of 2^p-th root selections

Just like in the field case, we want to define 2^p-th root selections as homomorphisms that assign to 2^p-th powers of a hyperfield H some elements of the multiplicative group of H. A word of explanation is needed here, as we have to clarify what we mean by homomorphisms. For two hyperfields H_1 and H_2 there are at least two ways to define morphisms that lead to two distinct categories with substantially different properties: following [8], by a *strict homomorphism* we understand a function $\phi : H_1 \to H_2$ such that

(i) $\forall a, b \in H_1[\phi(a + b) = \phi(a) + \phi(b)]$,

(ii) $\phi(0) = 0$,

(iii) $\forall a \in H_1[\phi(-a) = -\phi(a)]$,

(iv) $\forall a, b \in H_1[\phi(a \cdot b) = \phi(a) \cdot \phi(b)]$,

(v) $\phi(1) = 1$,

and by *homomorphism* we understand a function $\phi : H_1 \to H_2$ that satisfies the axioms (ii)–(v). but such that the axiom (i). is replaced with

(i') $\forall a, b \in H_1[\phi(a + b) \subseteq \phi(a) + \phi(b)]$.

Nevertheless, since the multiplication in hyperfields is just the usual, single-valued binary operation, it follows that non-zero elements of a hyperfield form a usual multiplicative group, and no matter which definition of a morphism of hyperfields we consider, it always leads to the same definition of a most standard homomorphism of underlying multiplicative groups.

The existence of the abovementioned homomorphism that assigns to 2^p-th powers of a hyperfield H some elements of the multiplicative group of H is equivalent to the existence of a certain subgroup of H^*, as shown in the following lemma:

Lemma 1. *Let H be a hyperfield and assume that H contains the 2^p-th primitive root of unity ω_{2^p}. A multiplicative homomorphism ϕ from the group H^{*2^p} of nonzero 2^p-th powers of H to H^* such that $\phi(c^{2^p}) = \omega_{2^p}^k c$, for some $k \in \{1, \ldots, 2^p\}$, exists if and only if there exists a multiplicative subgroup R of H such that for every element $a \in H^*$ there exist a unique element $r \in R$ and a unique integer $k \in \{1, \ldots, 2^p\}$ such that $a = \omega_{2^p}^k r$.*

Proof. ($\Rightarrow$) Assume that there exists ϕ as required. Let $R = \mathrm{Im}\phi$. Clearly R is a subgroup of H^*. Fix an element $a \in H^*$. Thus, for some $k \in \{1, \ldots, 2^p\}$, $H \ni \phi(a^{2^p}) = \omega_{2^p}^k a$, and hence $H^* = \omega_{2^p} R \cup \omega_{2^p}^2 R \cup \cdots \cup R$. Say that, for some $r_1, r_2 \in R$ and some $k_1, k_2 \in \{1, \ldots, 2^p\}$, $\omega_{2^p}^{k_1} r_1 = \omega_{2^p}^{k_2} r_2$. We might as well assume that $k_2 \geq k_1$ and denote $l = k_2 - k_1$. Therefore $\omega_{2^p}^l \in R$, say $\omega_{2^p}^l = \phi(c^{2^p})$, for some $c \in H^*$. On the other hand $\phi(c^{2^p}) = \omega_{2^p}^k c$, for some $k \in \{1, \ldots, 2^p\}$, so that $c = \omega_{2^p}^{l-k}$. But then $c^{2^p} = (\omega_{2^p}^{2^p})^{l-k} = 1$. Since $1 \cdot 1 = 1$, it follows that $\phi(1) \cdot \phi(1) = \phi(1)$, and hence $\phi(1) = 1$. In particular, $\omega_{2^p}^l = 1$, so that, since $l < 2^p$, $l = 0$. As a result, $k_2 = k_1$ which leads to $r_1 = r_2$ as well.

($\Leftarrow$) Conversely, assume that there exists R as claimed. Then, for a fixed $c^{2^p} \in H^{*2^p}$, there are uniquely defined an element $r \in R$ and an integer $k \in \{1, \dots, 2^p\}$ such that $\omega_{2^p}^k c = r$. The assignment $\phi(c^{2^p}) := r$ declares a well-defined function that satisfies the desired condition and it remains to check that it is a group homomorphism: indeed, for if $c_1^{2^p}, c_2^{2^p} \in H^{*2^p}$ and $k_1, k_2 \in \{1, \dots, 2^p\}$, $r_1, r_2 \in R$ are the unique integers and elements of R such that $\omega_{2^p}^{k_1} c_1 = r_1$ and $\omega_{2^p}^{k_2} c_2 = r_2$, then $\omega_{2^p}^{k_1+k_2} c_1 c_2 = r_1 r_2$, which yields $\phi(c_1^{2^p} c_2^{2^p}) = r_1 r_2 = \phi(c_1^{2^p})\phi(c_2^{2^p})$. ∎

Corollary 1. *Let H be a hyperfield. A multiplicative homomorphism ϕ from the group H^{*2} of nonzero squares of H to H^* such that $\phi(c^2) = \pm c$ exists if and only if there exists a multiplicative subgroup R of H such that for every element $a \in H^*$ there exist a unique element $r \in R$ such that $a = \pm r$.*

Root selections and 2^p-th root selections can be now defined just like in the field case.

Definition 1. Let H be a hyperfield and assume that H contains the 2^p-th primitive root of unity ω_{2^p}. A multiplicative subgroup R of H^* such that for every element $a \in H^*$ there exist a unique element $r \in R$ and a unique integer $k \in \{1, \dots, 2^p\}$ such that $a = \omega_{2^p}^k r$ shall be called a *2^p-th root selection for H*. In case when $p = 1$ we shall simply call it a *root selection*.

The existence of 2^p-th root selections is granted by the following, slightly more general result:

Theorem 1. *Let H be a hyperfield and assume that H contains the 2^p-th primitive root of unity ω_{2^p}. Let $T \subset H^*$ be a set of nonzero elements of H. Then there exists a 2^p-th root selection for H containing T if and only if the subgroup $H^{*2^p}[T] < H^*$ generated by T and the group of all 2^p-th powers does not contain -1.*

Proof. ($\Rightarrow$) Assume that there exists a 2^p-th root selection R for H containing T. Assume, *a contrario,* that $-1 \in H^{*2^p}[T]$. Note that $H^{*2^p} \subset R$: indeed, for a fixed $c^{2^p} \in H^{*2^p}$, there exist $r \in R$ and $k \in \{1, \dots, 2^p\}$ such that $c = \omega_{2^p}^k r$, hence $c^{2^p} = (\omega_{2^p}^{2^p})^k r^{2^p} = r^{2^p} \in R$. Thus $-1 \in H^{*2^p}[T] \subset R$. But $\omega_{2^p}^{2^{p-1}} = -1$, so that $1 = \omega_{2^p}^{2^p} \cdot 1$ and $1 = \omega_{2^p}^{2^{p-1}} \cdot (-1)$, contrary to the uniqueness of the presentation of 1.

($\Leftarrow$) Assume that $-1 \notin H^{*2^p}[T]$. Let $\mathcal{S} = \{S \mid S < H^*, H^{*2^p}[T] \subset S, -1 \notin S\}$. Union of any chain $\mathcal{C}$ of elements of $\mathcal{S}$ is again an element of $\mathcal{S}$, so, by Zorn's Lemma, let R be a maximal element of $\mathcal{S}$. In order to show that R is a 2^p-th root selection for H, we first claim that for $a \in H^*$ such that $a^2 \in R$ either $a \in R$ or $-a \in R$. Indeed, suppose that neither $a \in R$ nor $-a \in R$, for some $a \in H^*$ such that $a^2 \in R$. $R \cup aR$ is easily seen to be a group, and $-1 \notin R \cup aR$, for if $-1 = ar$, for some $r \in R$, then $-a = a^2 r \in R$. But $R \subsetneq R \cup aR$ – a contradiction.

Fix $a \in H^*$. Then $a^{2^p} \in R$, so, by our claim, either $a^{2^{p-1}} \in R$ or $-a^{2^{p-1}} \in R$. If the latter is the case, as $-\omega_{2^p}^{2^{p-1}} = (-1) \cdot (-1) = 1 \in R$, consequently, $(a\omega_{2^p})^{2^{p-1}} = a^{2^{p-1}}\omega_{2^p}^{2^{p-1}} = (-a^{2^{p-1}}) \cdot (-\omega_{2^p}^{2^{p-1}}) \in R$. Repeating the argument, we get that either $a^{2^{p-2}} \in R$ or $(a\omega_{2^p}^l)^{2^{p-2}} \in R$, for some $l \in \{1, \ldots, 2^p - 1\}$. By induction, we eventually show that either $a \in R$ or $a\omega_{2^p}^k \in R$, for some $k \in \{1, \ldots, 2^p - 1\}$. ■

Corollary 2. *Let H be a hyperfield. Let $T \subset H^*$ be a set of nonzero elements of H. Then there exists a root selection for H containing T if and only if the subgroup $H^{*2}[T] < H^*$ generated by T and the group of nonzero squares does not contain -1.*

A necessary and sufficient condition for a 2^p-th root selection to exist now easily follows:

Theorem 2. *Let H be a hyperfield and assume that H contains the 2^p-th primitive root of unity ω_{2^p}. A 2^p-th root selection for H exists if and only if H does not contain a 2^p-th root of -1.*

Proof. In Theorem 1 take T to be empty. ■

Corollary 3. *Let H be a hyperfield. A root selection for H exists if and only if H does not contain a square root of -1.*

3. 2^p-TH ROOT SELECTIONS VS. ORDERINGS OF LEVEL p

At this point we shall outline some similarities between 2^p-th root selections and orderings of level p. A classical result due to Artin and Schreier states that an ordering exists in a field F if and only if -1 is not a sum of squares. A version of the same result for orderings of higher level due to Becker [3] says that an ordering of level p exists in F if and only if -1 is not a sum of 2^p-th powers. Corresponding theorems for hyperfields ensure that an ordering exists in a hyperfield H if and only if H is *formally real*, that is if -1 is not an element of the set of sums of squares ([11], Lemma 3.3), and that an ordering of level p exists in a hyperfield H if and only H is *formally p-real*, that is if -1 is not an element of the set of sums of 2^p-th powers ([5], Theorem 1 and [6], Theorem 4.1). Therefore, by Theorem 2, if a hyperfield H contains a 2^p-th primitive root of unity and admits an ordering of level p, then it also admits a 2^p-th root selection (or, by Corollary 3, if a hyperfield admits an ordering, then it also admits a root selection). We can, in fact, be a bit more specific; recall that a (pre-) ordering (of level p) P is *proper* if $-1 \notin P$, and that a proper ordering is a maximal proper preordering ([5], Remark 8 p. 19) – we then have:

Proposition 1. *Let H be a formally p-real hyperfield, let P be a proper ordering of exact level p. Then H contains the 2^p-th primitive root of unity ω_{2^p} and P^* is a 2^p-th root selection.*

Proof. By definition P^* is a subgroup of the multiplicative group H^*. Say ξP^* is a generator of the cyclic group H^*/P^* of order 2^p. Since ξP^* is the homomorphic image of the element ξ via the canonical group epimorphism $\kappa : H^* \to H^*/P^*$, the order $\mathrm{ord}(\xi)$ of ξ in H^* is divisible by 2^p, say $\mathrm{ord}(\xi) = 2^p m$. But then $(\xi^m)^{2^p} = 1$, and, clearly, $(\xi^m)^k \neq 1$, for $k \in \{1, \ldots, 2^p - 1\}$, so ξ^m is the 2^p-th primitive root of unity ω_{2^p}.

In order to show that for every element $a \in H^*$ there exist a unique element $r \in P^*$ and a unique integer $k \in \{1, \ldots, 2^p\}$ such that $a = \omega_{2^p}^k r$, we partially repeat the argument of Theorem 1: firstly we claim that for $a \in H^*$ such that $a^2 \in P^*$ either $a \in P^*$ or $-a \in P^*$, and this is, indeed, the case, for if neither $a \in P^*$ nor $-a \in P^*$, for some $a \in H^*$ with $a^2 \in P^*$, then $P \cup aP$ would be a proper preordering bigger than P: it is easily seen to be a preordering, and $-1 \notin P \cup aP$, for if $-1 = ar$, for some $r \in P$, then $-a = a^2 r \in P^*$.

Now fix $a \in H^*$. Then $a^{2^p} \in P^*$, so, by the claim, either $a^{2^{p-1}} \in P^*$ or $-a^{2^{p-1}} \in P^*$. If the latter is the case, as $-\omega_{2^p}^{2^{p-1}} \in P^*$ consequently, $(a\omega_{2^p})^{2^{p-1}} \in P^*$. Repeating the argument, we get that either $a^{2^{p-2}} \in P^*$ or $(a\omega_{2^p}^l)^{2^{p-2}} \in P^*$, for some $l \in \{1, \ldots, 2^p - 1\}$. Going down, we eventually show that either $a \in P^*$ or $a\omega_{2^p}^k \in P^*$, for some $k \in \{1, \ldots, 2^p - 1\}$. ∎

It is not hard to see that there exist (2^p-th) root selections that do not come from orderings (of level p) in hyperfields: since every field can be seen as a hyperfield with addition defined by $a + b = \{a + b\}$, examples of such root selections (in particular examples of Section 2 in [12]) presented in earlier papers on the theme remain valid in our setting. We shall, however, add one more example to this catalogue and show that a root selection in a field F, $\mathrm{char}F \neq 2$, $F \neq \mathbb{F}_3, \mathbb{F}_5$, defines a root selection in the quadratic hyperfield $Q(F)$:

Example 1. Let F be a field, $\mathrm{char}F \neq 2$, $F \neq \mathbb{F}_3, \mathbb{F}_5$, let R be a root selection. Then R/F^{*2} is a root selection in $Q(F)$.

To see that this is the case, note that, since every element $a \in F^*$ can be uniquely presented as $a = \pm r$ for some $r \in R$, it is clear that every element $aF^{*2} \in Q(F)$ can be presented as $aF^{*2} = \pm rF^{*2}$. Suppose that, for some $a \in F^*$, we have two such presentations, $aF^{*2} = \epsilon_1 r_1 F^{*2} = \epsilon_2 r_2 F^{*2}$, for some $\epsilon_1, \epsilon_2 \in \{+1, -1\}$, $r_1, r_2 \in R$. But then $a = \epsilon_1 r_1 s^2$ and $a = \epsilon_2 r_2 t^2$, for some $s, t \in F^*$. Since $s^2 \in R$ (if $s = \pm r$, for some $r \in R$, then $s^2 = r^2 \in R$, as R is a group), $r_1 s^2 \in R$ and, similarly, $r_2 t^2 \in R$. Thus $\epsilon_1 = \epsilon_2$ and $r_1 s^2 = r_2 t^2$ meaning $r_1 F^{*2} = r_2 F^{*2}$.

We urge the reader who is familiar with general quadratic hyperfields $Q(F)$ (without the additional assumptions $\mathrm{char}F \neq 2$, $F \neq \mathbb{F}_3, \mathbb{F}_5$) to check that the example remains valid in this slightly more general setting.

Observe that, since the multiplicative group of $Q(F)$ is of exponent 2, it does not make much sense to consider 2^p-th root selections in quadratic hyperfields with $p > 1$. At the same time Example 1 is quite nice in the sense that a corresponding result also holds for orderings, that is an ordering of a field F leads to an ordering of $Q(F)$: indeed, suppose that F has an ordering, which is equivalent to -1 not being a sum of squares, and that $Q(F)$ is not formally real. Then, by Lemma 3.3 of [11], $-F^{*2}$ is a sum of squares in $Q(F)$. Since $(aF^{*2})^2 = F^{*2}$ in $Q(F)$, this simply means that $-F^{*2} = F^{*2} + \cdots + F^{*2}$ in $Q(F)$, which means that $-1 \in D(1,\ldots,1)$, which is equivalent to say that -1 is a sum of squares – a contradiction.

The next analogy between (2^p-th) root selections and orderings (of level p) in hyperfields is captured by the following theorem, which describes when an element of a hyperfield can be incorporated in a (2^p-th) root selection:

Theorem 3. *Let H be a hyperfield and assume that H contains the 2^p-th primitive root of unity ω_{2^p}. Let $a \in H^*$ and assume that H does not contain a 2^p-th root of -1. Then there exists a 2^p-th root selection R such that $a \in R$ if and only if $-a^k \notin H^{*2^p}$, for all $k \in \{1,\ldots,2^p-1\}$.*

Proof. In Theorem 1 take $T = \{a\}$. Then $-1 \notin H^{*2^p}[a]$ if and only if $-1 \neq c^{2^p}a^k$, for some $c \in H^*$ and $k \in \{1,\ldots,2^p-1\}$, or, equivalently, $-a^l \notin H^{*2^p}$, for some $l \in \{1,\ldots,2^p-1\}$. ■

Corollary 4. *Let H be a hyperfield. Let $a \in H^*$ and assume that H does not contain a square root of -1. Then there exists a root selection R such that $a \in R$ if and only if $-a \notin H^{*2}$.*

Finally, as a version of the classical result by Artin and Schreier, it is known that the intersection of all orderings (of level p) is the preordering (of level p) consisting of all sums of squares (sums of 2^p-th powers): see [11] Proposition 3.4, [5] Corollary 3 and [6] Theorem 4.2. The corresponding result for (2^p-th) root selections is the following one:

Theorem 4. *Let H be a hyperfield and assume that H contains the 2^p-th primitive root of unity ω_{2^p}. Let $a \in H^*$ and assume that H does not contain a 2^p-th root of -1. Then a belongs to all 2^p-th root selections in H if and only if $a \in H^{*2^p}$.*

Corollary 5. *Let H be a hyperfield. Let $a \in H^*$ and assume that H does not contain a root of -1. Then a belongs to all root selections in H if and only if $a \in H^{*2}$.*

Remark 1. For formally real fields one introduces the notion of a *real closure*, which is a maximal algebraic extension of a formally real field which is still formally real, and proceeds to show that such a real closed field is always uniquely ordered by the set of all squares. A corresponding result also holds for formally p-real fields [3, 10]. Further, one introduces the notion of a *maximal root selection* field, which is a maximal algebraic extension of a field equipped with a root selection which also admits a root selection, and it is possible to show that such maximal root selection always exists, and its unique root selection consists of squares ([12], Theorems 3 and 4). A corresponding result exists also for 2^p-th root selections in fields ([4], Theorems 3.4 and 4.1). Unfortunately, this part of the theory can not be carried to the hyperfield case: the theory of hyperfield extensions does not seem to be well developed, in particular there is no satisfactory notion of an algebraic hyperfield extension, yet alone of a usable Galois theory.

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