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A Semantic Approach to Conservativity

Abstract. The aim of this paper is to describe from a semantic perspective the problem of conservativity of classical first-order theories over their intuitionistic counterparts. In particular, we describe a class of formulae for which such conservativity results can be proven in case of any intuitionistic theory T which is complete with respect to a class of T-normal Kripke models. We also prove conservativity results for intuitionistic theories which are closed under the Friedman translation and complete with respect to a class of conversely well-founded Kripke models. The results can be applied to a wide class of intuitionistic theories and can be viewed as generalization of the results obtained by syntactic methods.

Keywords: Classical and intuitionistic first order theories, Conservativity, Kripke models.

1. Introduction

Let Γ be a class of formulae of some first-order language. We say that a classical theory is conservative over its intuitionistic counter-part with respect to Γ if both theories prove exactly the same formulae of this class. A typical example of a conservativity result states that Peano Arithmetic (PA) is Π_2-conservative over Heyting Arithmetic (HA). It can be proven in several ways. For example, the so-called (Gödel-Gentzen) negative translation together with the Gödel functional interpretation of HA or proof theoretic analysis of HA can be used. This fact can be also proven by means of the negative translation and the so-called Friedman translation. This approach can be applied also to other theories including set theory. More recently, new methods for proving conservativity were developed by T. Coquand and M. Hofmann in [5] and J. Avigad in [2]. A generalization of conservativity theorem for PA and extensions of HA by adding restricted versions of the Law of Excluded Middle was considered in [3]. All of the above papers present a syntactic approach.

The aim of this paper is to describe conservativity of classical first-order theories over their intuitionistic counterparts from a semantic perspective.

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In particular, we consider properties of a class of Kripke models for a given intuitionistic theory that are sufficient to prove conservativity results. We also describe a class of formulae for which such results can be proven. The paper provides applications of main results.

2. Preliminaries

Let L be a fixed first-order language with logical symbols $\bot$, $\wedge$, $\vee$, $\rightarrow$ and quantifiers $\forall$ and $\exists$. We consider a set T of sentences of L viewed as a set of axioms. Then T gives rise to a classical theory T^{c}, when closed under consequences of classical logic, or an intuitionistic theory T^{i}, when closed under consequences of intuitionistic logic. Formally, the theories T^{c} and T^{i} are defined as $\mathsf{T}^c = \{A : \mathsf{T} \vdash^c A\}$ and $\mathsf{T}^i = \{A : \mathsf{T} \vdash^i A\}$, where $\vdash^c$ and $\vdash^i$ are classical and intuitionistic consequence relations respectively. Sometimes we will write also $\mathsf{T}^{\mathsf{c}} \vdash A$ and $\mathsf{T}^{\mathsf{i}} \vdash A$ instead of $\mathsf{T} \vdash^c A$ and $\mathsf{T} \vdash^i A$. Let us note that we say that a formula A is *decidable* in an intuitionistic theory T^{i} if $\mathsf{T}^{\mathsf{i}} \vdash A \vee \neg A$.

As it was mentioned, in proving conservativity results, the syntactic translations play an important role. The *negative translation* assigns to each formula A the formula A^- in such a way that for subformulae B and C of A, we have $B^- = \neg\neg B$ when B is atomic, $(B \vee C)^- = \neg(\neg B^- \wedge \neg C^-)$ and $(\exists x B)^- = \neg\forall x \neg B^-$; the negative translation commutes with conjunction, implication and universal quantifier. Obviously, in classical logic, A^- is equivalent to A for all formulae A. It is known that a sentence A is *classically* provable from T then its negative translation A^- is also *intuitionistically* provable from the set of negative translations of the formulae from T. The *Friedman translation* was introduced in [7] to study conservativity of arithmetic and set theory. Let us recall that, for a fixed formula F, the F-Friedman translation is defined as follows: to any formula A (where no free variable of F is quantified in A, otherwise we rename the free variables of F), we assign the formula A^F obtained from A by replacing each atomic subformula B of A with $B \vee F$. In classical logic, A^F is equivalent to $A \vee F$ for all formulae A and F. The main feature of the Friedman translation states that if a formula is intuitionistically derivable from T then its F-translation A^F is also intuitionistically derivable from the set T^F of F-translations of the formulae of T. We say that an intuitionistic theory T^{i} is closed under the given translation, if T^{i} proves the translations of all its axioms. In [7] H. Friedman used a combination of the two translations to prove that PA is Π_2-conservative over HA. Here we rephrase this result in the following generalized form.

THEOREM 2.1. *Let* T^i *be an intuitionistic theory closed under the Friedman and the negative translation and such that all atomic formulae are decidable in* T^i*. Then* T^c *is* $\forall\exists$*-conservative over* T^i*.*

The proof of Theorem 2.1 consists of two parts. First, we apply the negative translation to interpret T^c within T^i. Then, we use the Friedman translation to restore the previous constructive meaning of the formulae translated in the first step.

Let us observe that application of the negative translation affects the meaning of the translated formula since all disjunctions and existential [4] quantifiers are eliminated according to De Morgan's Laws. In particular, the translation of an axiom of a theory T^i need not be provable in T^i anymore. For example, in the fragment $\mathsf{i}\Sigma_1$ of intuitionistic arithmetic in which the induction schema is restricted to Σ_1-formulae only, the negative translation of an Σ_1-induction axiom is not provable in $\mathsf{i}\Sigma_1$. Notice also that, in general, the Friedman translation increases the complexity of translated formulae. So, subtheories of intuitionistic arithmetic, such as $\mathsf{i}\Delta_0$ are not closed under the Friedman translation. Thus the method of proving conservativity by means of the negative and Friedman translations cannot be applied in such cases.

In this paper we focus on Kripke semantics for intuitionistic first-order theories. In the context of conservativity, this choice seems to be natural, since in this case we can observe an interplay between intuitionistic and classical theories, and interplay between their models.

Now we recall basic notions and facts concerning Kripke models which will be needed in the sequel. By a *Kripke model* $\mathcal{M}$ for the language L we mean a tuple

$$\mathcal{M} = (W, \leq, \{\mathbf{M}_w : w \in W\})$$

where W is a non-empty set of *nodes* partially ordered by $\leq$, and for every w, $\mathbf{M}_w$ is a classical first-order structure for L called a *world* of the model $\mathcal{M}$. We assume that if $w \leq v$ then $\mathbf{M}_w$ is a weak substructure of $\mathbf{M}_v$, i.e. the domain M_w of the structure $\mathbf{M}_w$ is a subset of the domain M_v of $\mathbf{M}_v$ and for all atomic formulae $P(x_1, \dots, x_n)$ and all $a_1, \dots, a_n \in \mathrm{M}_w$,

$$\text{if } \ \mathbf{M}_w \models P(a_1, \dots, a_n) \ \text{ then } \ \mathbf{M}_v \models P(a_1, \dots, a_n).$$

The pair $(W, \leq)$ is called the *frame* of the model $\mathcal{M}$.

The intuitionistic forcing relation $\Vdash$ in the model $\mathcal{M}$ is defined in terms of the classical satisfaction relation $\models$ considered locally for the worlds. More precisely, for a formula $A(\overline{a})$, with parameters $\overline{a} = a_1, \dots, a_n$ from the world

M_w, forcing of $A(\overline{a})$ at the node w of the Kripke model $\mathcal{M}$ is defined in the following way.

The constant *falsum* is not forced at any node

- $(\mathcal{M}, w) \nVdash \bot$.

If $A(\overline{a})$ is an atomic formula, then

- $(\mathcal{M}, w) \Vdash A(\overline{a})$ iff $\mathbf{M}_w \models A(\overline{a})$.

Then forcing is extended inductively to all formulae in the following way (for better readability, we omit unnecessary occurrences of variables and parameters):

- $(\mathcal{M}, w) \Vdash A \wedge B$ iff $(\mathcal{M}, w) \Vdash A$ and $(\mathcal{M}, w) \Vdash B$
- $(\mathcal{M}, w) \Vdash A \vee B$ iff $(\mathcal{M}, w) \Vdash A$ or $(\mathcal{M}, w) \Vdash B$
- $(\mathcal{M}, w) \Vdash A \rightarrow B$ iff for all $v \geq w$ we have $(\mathcal{M}, v) \nVdash A$ or $(\mathcal{M}, v) \Vdash B$
- $(\mathcal{M}, w) \Vdash \exists x A(x)$ iff there is $a \in \mathrm{M}_w$ such that $(\mathcal{M}, w) \Vdash A(a)$
- $(\mathcal{M}, w) \Vdash \forall x A(x)$ iff for every $v \geq w$ and for every $a \in \mathrm{M}_v$ we have $(\mathcal{M}, v) \Vdash A(a)$.

The forcing is monotone with respect to all formulae, i.e. if $(\mathcal{M}, w) \Vdash A$ and $w \leq v$ then $(\mathcal{M}, v) \Vdash A$.

As usual, we say that a formula $A(\bar{x})$ is *valid* in $\mathcal{M}$, which is denoted by $\mathcal{M} \Vdash A(\bar{x})$, if $(\mathcal{M}, w) \Vdash A(\bar{a})$ for any node w and any tuple of parameters $\bar{a}$ from M_w. If the context is clear and the model $\mathcal{M}$ is fixed, we will write simply $w \Vdash A$ instead of $(\mathcal{M}, w) \Vdash A$.

Let us note that, in general, the relation between classical and intuitionistic validity in a Kripke model $\mathcal{M}$ cannot be easily described. In particular, the classical validity of a formula A at a world $\mathbf{M}_w$ coincides with intuitionistic forcing of A at the corresponding node w in the model $\mathcal{M}$ only for formulae built up from atoms, conjunction, disjunction and existential quantifier. Moreover, there is no straightforward correlation between the (intuitionistic) theory of the Kripke model and the (classical) theories of its worlds.

3. The Idea

In order to prove classically that a theory T^{c} is Γ-conservative over its intuitionistic counterpart T^{i}, we may show that any formula from Γ which is not derivable intuitionistically in T^{i} is also not derivable classically in T^{c}.

From semantic point of view it means that if there is a model of T^{i} which is a counter-model of A, the we can find a model of T^{c}, the classical counterpart of T^{i}, which is a counter-model of A. Our idea is to use Kripke semantics for intuitionistic first order logic, and look for a suitable classical counter-model of A among the worlds of a Kripke model which refutes A.

More precisely, assume that $A \in \Gamma$ and $\mathsf{T}^{\mathsf{i}} \nvdash A$. So, by the strong completeness theorem for Kripke semantics, we can find a Kripke model $\mathcal{M}$ of T^{i} such that $\mathcal{M}$ refutes A. Thus, since $\mathcal{M} \nVdash A$, we can find a node w of $\mathcal{M}$, such that $w \nVdash A$. In general, the world $\mathbf{M}_w$ corresponding to the node w in $\mathcal{M}$ need not be a counter-model for A nor a model of T^{c}. However, it is enough to find *some* node u such that $\mathbf{M}_u \not\models A$ and $\mathbf{M}_u \models \mathsf{T}^{\mathsf{c}}$, for the world $\mathbf{M}_u$ corresponding to u in $\mathcal{M}$. We show that under suitable assumptions concerning models of the theory T^{i} (and some assumptions on T^{i} itself) this can be done.

The first case we consider in this paper is that when all the worlds of the model $\mathcal{M}$ in question are classical models of the theory T^{c}. In this case, the model in question is called T^{c}-normal, see [4], and [6] where this notion was implicitly introduced for the first time.

But even if we do not know whether a Kripke model $\mathcal{M}$ of a theory T^{i} is T^{c}-normal, we can sometimes prove that it contains a world which is a model of the theory T^{c}. We can show it using the method of pruning introduced by D. van Dalen, H. Mulder, E.C.W. Krabbe and A. Visser in [6]. Essentially, Kripke models of intuitionistic theories which are closed under the Friedman translation admit pruning.

In the following two sections we present details concerning both of the methods sketched above.

4. Conservativity and T^{c}-Normal Kripke Models

In this section we focus on T^{c}-normal Kripke models and intuitionistic theories which are complete with respect to a class of T^{c}-normal Kripke models. We begin with recalling the following definition from [4].

Definition 4.1. Let T be a set of first-order sentences. A Kripke model $\mathcal{M} = (W, \leq, \{\mathbf{M}_w : w \in W\})$ is called T^{c}-*normal* if for every $w \in W$, we have $\mathbf{M}_w \models \mathsf{T}^{\mathsf{c}}$, i.e. each world $\mathbf{M}_w$ of $\mathcal{M}$ is a model of the classical theory T^{c}.

Let us note that not every T^{c}-normal Kripke model is a model of the intuitionistic theory T^{i}. Also, a Kripke model of a theory T^{i} need not be

T^{c}-normal. We say that an intuitionistic theory T^{i} is *complete with respect to the class of* T^{c}*-normal models* iff whenever a sentence A is true in every T^{c}-normal model then A is provable in T^{i}.

Now we introduce a class of formulae which will be used in the sequel.

DEFINITION 4.2. We say that a formula A is *forcing-stable (f-stable* for short*) in a theory* T^{i} iff for every Kripke model $\mathcal{M}$ of T^{i} and every node w in $\mathcal{M}$ we have

$$\text{if } w \Vdash A \text{ then } \mathbf{M}_w \models A.$$

A formula A is *stable in a theory* T^{i} iff for every Kripke model $\mathcal{M}$ of T^{i} and every node w in $\mathcal{M}$ we have

$$w \Vdash A \text{ iff } \mathbf{M}_w \models A.$$

It is well-known that the class of formulae which are stable in IQC coincides with the set of positive formulae, i.e. the formulae built up from atoms, by means of disjunction, conjunction and existential quantification only, see [9]. Here we need a more general class of formulae possibly extending the class of positive formulae.

DEFINITION 4.3. Let $\mathcal{P}(\mathsf{T}^{\mathsf{i}})$ be the smallest class such that $\mathcal{P}(\mathsf{T}^{\mathsf{i}})$ contains all atomic formulae, all formulae which are decidable in T^{i} and $\mathcal{P}(\mathsf{T}^{\mathsf{i}})$ is closed under disjunctions, conjunctions and existential quantification.

Let us note the following fact.

PROPOSITION 4.4. *For every theory* T^{i}*, if* $A \in \mathcal{P}(\mathsf{T}^{i})$ *then* A *is stable.*

PROOF. An induction on the complexity of a formula. ∎

Recall that a formula A is called *semi-positive* if each subformula of A of the form $B \to C$ has B atomic. Note that the class of semi-positive formulae is exactly the class of formulae which are preserved under taking submodels of Kripke models resulting in restricting the frame of a given model, see [12]. Of course, classically every first-order formula is equivalent to a semi-positive formula. Note that the semi-positive formulae are f-stable in any theory T^{i}. Also it is easy to see that if T^{i} is an intuitionistic theory in which all atomic formulae are decidable then every prenex formula is forcing-stable in T^{i}.

We define a class of formulae which generalizes the class of semi-positive formulae and show that it is contained in the class of f-stable formulae in a theory T^{i}.

DEFINITION 4.5. Let T^{i} be an intuitionistic theory. The class $\mathcal{G}(\mathsf{T}^{\mathsf{i}})$ of *generalized semi-positive formulae in* T^{i} is the least class of formulae such that

(i) $\bot \in \mathcal{G}(\mathsf{T}^{\mathsf{i}})$,

(ii) $\mathcal{P}(\mathsf{T}^{\mathsf{i}}) \subseteq \mathcal{G}(\mathsf{T}^{\mathsf{i}})$,

(iii) if $B, C \in \mathcal{G}(\mathsf{T}^{\mathsf{i}})$ then $B \wedge C, B \vee C, \exists x B, \forall x B \in \mathcal{G}(\mathsf{T}^{\mathsf{i}})$,

(iv) if $B \in \mathcal{P}(\mathsf{T}^{\mathsf{i}})$ and $C \in \mathcal{G}(\mathsf{T}^{\mathsf{i}})$, then $B \to C \in \mathcal{G}(\mathsf{T}^{\mathsf{i}})$.

In particular, if in T^{i} all atomic formulae are decidable, then every formula of the form $\exists x A$, where A is a quantifier-free formula, belongs to $\mathcal{P}(\mathsf{T}^{\mathsf{i}})$. In consequence, all prenex formulae are members of $\mathcal{G}(\mathsf{T}^{\mathsf{i}})$.

We will show that the class of generalized semi-positive formulae has the desired property.

Proposition 4.6. *For every intuitionistic theory T^{i}, every generalized semi-positive formula in T^{i} is forcing-stable in T^{i}.*

Proof. Let T^{i} be an intuitionistic theory. It is clear that $\bot$ and every formula from $\mathcal{P}(\mathsf{T}^{\mathsf{i}})$ is f-stable. Moreover, it is easy to see that the class of f-stable formulae is closed under disjunction, conjunction and quantification. So, only the condition (iv) is to be checked.

Let us fix a model $\mathcal{M}$ and a node w of $\mathcal{M}$ and consider a formula $A \to B$ where $A \in \mathcal{P}(\mathsf{T}^{\mathsf{i}})$ and $B \in \mathcal{G}(\mathsf{T}^{\mathsf{i}})$. First, assume that $B \neq \bot$ and $w \Vdash A \to B$. Then, in particular, $w \nVdash A$ or $w \Vdash B$. In the former case, $\mathbf{M}_w \not\models A$ since $A \in \mathcal{P}(\mathsf{T}^{\mathsf{i}})$. In the latter case, by inductive hypothesis for B, we have $\mathbf{M}_w \models B$, and consequently $\mathbf{M}_w \models A \to B$. Assume that $B = \bot$, so $A \to B$ is equivalent to $\neg A$. Since A is stable, whenever $w \nVdash A$ then $\mathbf{M}_w \not\models A$. Hence, in particular, if $w \Vdash \neg A$ then $\mathbf{M}_w \models \neg A$. ∎

Let us note the following direct consequence of Proposition 4.6.

Corollary 4.7. *If T is a set of semi-positive sentences, then every Kripke model of T^{i} is T^{c}-normal. In particular, T^{i} is complete with respect to a class of T^{c}-normal Kripke models.*

We will need the following fact.

Lemma 4.8. *Assume that T^{i} is an intuitionistic theory and $\mathcal{M}$ is a Kripke model such that $\mathcal{M} \Vdash \mathsf{T}^{i}$. Then, for every formula $D(\bar{x}, y) \in \mathcal{P}(\mathsf{T}^{i})$ and for every world w of $\mathcal{M}$, the following conditions are equivalent*

(i) $w \Vdash \forall y D(\bar{a}, y)$

(ii) $\mathbf{M}_v \models \forall y D(\bar{a}, y)$ *for all* $v \geq w$,

where $\bar{a}$ are parameters from M_w.

Proof. The condition (i) implies (ii) by Proposition 4.4 and properties of forcing. For the converse implication, assume that $w \nVdash \forall y D(\bar{a}, y)$. This happens if and only if there is $v \geq w$ such that $v \nVdash D(\bar{a}, b)$ for some element b of M_v. Since $D \in \mathcal{P}(\mathsf{T}^{\mathsf{i}})$, we get $\mathbf{M}_v \not\models D(\bar{a}, b)$ and, consequently, $\mathbf{M}_v \not\models \forall y \exists z D(\bar{a}, y)$. ■

Let us define a class of formulae.

Definition 4.9. For a given theory T^{i} we define

$$\mathcal{A}(\mathsf{T}^{\mathsf{i}}) = \{\forall \bar{x}\big(C(\bar{x}) \to \forall y D(\bar{x}, y)\big) : C \in \mathcal{G}(\mathsf{T}^{\mathsf{i}}) \text{ and } D \in \mathcal{P}(\mathsf{T}^{\mathsf{i}})\}.$$

We can prove the main result of this section.

Theorem 4.10. *Assume that T^{i} is complete with respect to a class of T^{c}-normal Kripke models. Then T^{c} is conservative over T^{i} with respect to the class $\mathcal{A}(\mathsf{T}^{i})$.*

Proof. Let us fix a theory T^{i} as above, and a formula $C(\bar{x}) \to \forall y D(\bar{x}, y)$, where $C \in \mathcal{G}(\mathsf{T}^{\mathsf{i}})$ and $D \in \mathcal{P}(\mathsf{T}^{\mathsf{i}})$. Assume that $\mathsf{T}^{\mathsf{i}} \nvdash C(\bar{x}) \to \forall y D(\bar{x}, y)$. Then we can find a T^{c}-normal Kripke model $\mathcal{M}$ and a node w of $\mathcal{M}$ such that

$$w \Vdash C(\bar{a}) \tag{1}$$

and

$$w \nVdash \forall y D(\bar{a}, y), \tag{2}$$

for some sequence $\bar{a}$ of elements of M_w. From (2), by Lemma 4.8, there is $v \geq w$ such that

$$\mathbf{M}_v \not\models \forall y D(\bar{a}, y).$$

On the other hand, (1) implies that $v \Vdash C(\bar{a})$ and hence, since C is forcing-stable, we get

$$\mathbf{M}_v \models C(\bar{a}).$$

Hence

$$\mathbf{M}_v \not\models C(\bar{x}) \to \forall y D(\bar{x}, y).$$

Finally, $\mathbf{M}_v \models \mathsf{T}^{\mathsf{c}}$ since $\mathcal{M}$ is T^{c}-normal, and so we have a desired counter-model for the formula in question. Thus, by completeness theorem for T^{c}, we get $\mathsf{T}^{\mathsf{c}} \nvdash \forall \bar{x}\big(C(\bar{x}) \to \forall y D(\bar{x}, y)\big)$. ■

We can state the following direct consequences of Theorem 4.10. Note that Corollary 4.11 and Corollary 4.12 hold for IQC and all theories that are axiomatized by a set of generalized semi-positive axioms.

Corollary 4.11. *Let the theory T^{i} be complete with respect to a class of T^{c}-normal Kripke models. Then T^{c} is conservative over T^{i} with respect to the class $\{\forall x A : A \in \mathcal{P}(\mathsf{T}^{i})\}$.*

It is known that, although HA is not sound with respect to the class of PA-normal Kripke models, HA is complete with respect to it. This fact follows from the properties of axiomatization of the class of T^{c}-normal Kripke models and was proven in [4]. For details, see also [13]. Now, since in HA all atomic formulae are decidable, $\mathcal{P}(\mathsf{HA})$ is exactly the set of Σ_1 formulae of the language of arithmetic. Hence Π_2-conservativity of PA over HA follows.

We can also consider fragments of arithmetic that are not closed under syntactic translations. Recall that $\mathsf{i}\Delta_0$ is the fragment of HA in which induction is restricted to bounded formulae only. By a direct verification we can check that $\mathcal{M} \Vdash \mathsf{i}\Delta_0$ iff $\mathcal{M}$ is $\mathsf{I}\Delta_0$-normal, for every Kripke model $\mathcal{M}$. So, in particular, $\mathsf{i}\Delta_0$ is complete with respect to $\mathsf{I}\Delta_0$-normal models, where $\mathsf{I}\Delta_0$ is the classical counterpart of $\mathsf{i}\Delta_0$. Since atomic formulae are decidable in $\mathsf{i}\Delta_0$, we get Π_2 conservativity of $\mathsf{I}\Delta_0$ over $\mathsf{i}\Delta_0$.

Corollary 4.12. *Let the theory T^{i} be complete with respect to a class of T^{c}-normal Kripke models. Then T^{c} is conservative over T^{i} with respect to the class $\{\neg A : A \in \mathcal{G}(\mathsf{T}^{i})\}$.*

Note that the class $\{\forall x A : A \in \mathcal{P}(\mathsf{T}^{\mathsf{i}})\}$ which occurs in Corollary 4.11 can be slightly enlarged by all intuitionistic consequences of formulae classically equivalent to positive ones. The reason is that we only need to know that A is forced at the node w whenever A is satisfied in some world $\mathbf{M}_w$. The negations of generalized semi-positive formulae that occur in Corollary 4.12 have also this property.

5. Conservativity and Pruning

The technique of pruning was introduced in [6] to prove that all finite models of HA are PA-normal. We recall here from this paper the definition of pruning and the most important fact.

Definition 5.1. Let $\mathcal{M} = (W, \leq, \{\mathbf{M}_w : w \in W\})$ be a Kripke model and let $w \in W$. Assume that F is a sentence, possible with parameters from M_w, such that $(\mathcal{M}, w) \nVdash F$. We define the Kripke model

$$\mathcal{M}^F = (W^F, \leq^F, \{\mathbf{M}_v : v \in W^F\})$$

such that $W^F = \{v \in W : v \geq w \text{ and } (\mathcal{M}, v) \nVdash F\}$ and $\leq^F$ is the restriction of $\leq$ to the set W^F. The forcing relation of the model $\mathcal{M}^F$ is denoted by $\Vdash^F$.

The key result concerning the method of pruning is the following.

Lemma 5.2. (First Pruning Lemma) *Let $\mathcal{M}$ be a Kripke model and w be a node of $\mathcal{M}$ such that $(\mathcal{M}, w) \nVdash F$ for some sentence F with parameters from* M_w. *Then*

$$(\mathcal{M}, w) \Vdash A^F \quad \textit{iff} \quad (\mathcal{M}^F, w) \Vdash^F A,$$

for every A.

Let us state other properties of the Friedman translation which will be used in the sequel.

Lemma 5.3. *Let us fix a formula F. Then, for every formula A,*

(1) $\vdash^i F \to A^F$,

(2) *if A is atomic or decidable in* T *then* $\mathsf{T} \vdash^i (\exists x A)^F \leftrightarrow (\exists x A \vee F)$,

(3) *if* $\mathsf{T} \vdash^i A$ *then* $\mathsf{T}^F \vdash^i A^F$, *where* $\mathsf{T}^F = \{B^F : B \in \mathsf{T}\}$.

As a consequence of the above we get that if an intuitionistic theory T^{i} is closed under the F-Friedman translation then, whenever $\mathsf{T}^{\mathsf{i}} \vdash A$, we have also $\mathsf{T}^{\mathsf{i}} \vdash A^F$ for every formula A.

Definition 5.4. A formula is called *positive quantifier-free* if it is built from atoms by means of disjunctions and conjunctions only. Formula of the form $\forall x \exists y A$, where A is a positive formula is called a *positive* $\forall\exists$*-formula.*

We note the following variant of Friedman's lemma.

Lemma 5.5. *Consider a theory* T^{i}. *Let a formula A be positive or decidable in* T^{i}. *Then*

$$\mathsf{T}^{i} \vdash \exists x A^{\exists x A} \to \exists x A.$$

Moreover, if additionally the theory T^{i} *satisfies the formula*

$$\mathtt{CD} = \forall x (C(x) \vee D) \to (\forall x C(x) \vee D),$$

where the variable x is not free in D, then for any sequence of quantifiers Q_i

$$\mathsf{T}^{i} \vdash Q_1 x_1 \ldots Q_n x_n A^{Q_1 x_1 \ldots Q_n x_n A} \to Q_1 x_1 \ldots Q_n x_n A.$$

Proof. It is easy to check, by the complexity of the formula A, that for any formula A satisfying each of the assumptions of the theorem and for any formula F we have

$$\mathsf{T} \vdash A^F \leftrightarrow (A \vee F). \tag{3}$$

Now let us consider the formula $Q_n x_n \dots Q_1 x_1 A$, where Q_i are alternating quantifiers and put

$$F = Q_n x_n \dots Q_1 x_1 A(x_1, \dots, x_n).$$

Assume that

$$\mathsf{T}^{\mathsf{i}} \vdash (Q_n x_n \dots Q_1 x_1 A(\bar{x}))^F.$$

Then, by (3)

$$\mathsf{T}^{\mathsf{i}} \vdash Q_n x_n \dots Q_1 x_1 \big(A(\bar{x}) \vee F\big).$$

Whence we get

$$\mathsf{T}^{\mathsf{i}} \vdash Q_n x_n \dots Q_2 x_2 \big(Q_1 x_1 A(\bar{x}) \vee F\big),$$

by IQC when $Q_1 = \exists$, and by CD when $Q_1 = \forall$. Finally, after n steps, we get

$$\mathsf{T}^{\mathsf{i}} \vdash Q_n x_n \dots Q_1 x_1 A(\bar{x}) \vee Q_n x_n \dots Q_1 x_1 A(\bar{x}),$$

and consequently, $\mathsf{T}^{\mathsf{i}} \vdash Q_n x_n \dots Q_1 x_1 A(\bar{x})$. ■

The next results presented in this section concern theories that are closed under the Friedman translation.

Recall that $\mathcal{P}(\mathsf{T}^{\mathsf{i}})$ is the smallest class that contains all atomic formulae, all formulae which are decidable in T^{i} and $\mathcal{P}(\mathsf{T}^{\mathsf{i}})$ is closed under disjunctions, conjunctions and existential quantification.

Theorem 5.6. *Assume that the theory T^{i} is closed under the Friedman translation and complete with respect to a class of conversely well-founded Kripke models. Then T^{c} is conservative over T^{i} with respect to the class of formulae of the form $\forall x \exists y A$ where A belongs to $\mathcal{P}(\mathsf{T}^{i})$.*

Proof. Let A be a formula as in the assumption of the Theorem and such that $\mathsf{T}^{\mathsf{i}} \nvdash \forall x \exists y A$. Then we find a conversely well-founded Kripke model $\mathcal{M}$ of T^{i} such that $\mathcal{M} \nVdash \forall x \exists y A$. In particular, there is a node w of $\mathcal{M}$ and an element $a \in \mathrm{M}_w$ such that

$$(\mathcal{M}, w) \nVdash \exists y A(a, y).$$

By Lemma 5.5,

$$(\mathcal{M}, w) \nVdash \exists y A(a, y)^{\exists y A(a,y)}.$$

We prune the model $\mathcal{M}$ with respect to the formula $F := \exists y A(a, y)$. By the Pruning Lemma, in the pruned model $\mathcal{M}^F$ we have

$$(\mathcal{M}^F, w) \nVdash^F \exists y A(a, y).$$

Since the model $\mathcal{M}$ is conversely well-founded, there is a maximum world $v \geq w$ in $\mathcal{M}^F$. We have

$$(\mathcal{M}^F, v) \nVdash^F \exists y A(a, y) \quad \text{and} \quad (\mathcal{M}^F, v) \Vdash^F \mathsf{T}^{\mathsf{i}},$$

since T^{i} is closed under the Friedman translation. Moreover, since v is a terminal node in $\mathcal{M}^F$, we get

$$\mathbf{M}_v \not\models \forall x \exists y A(x, y) \quad \text{and } \mathbf{M}_v \models \mathsf{T}^{\mathsf{c}}$$

and hence, we get a desired counter-model. In particular,

$$\mathsf{T}^{\mathsf{c}} \nvdash \forall x \exists y A(x, y),$$

as required. ■

THEOREM 5.7. *Assume that the theory* T^{i} *is closed under the Friedman translation and complete with respect to the class of conversely well-founded Kripke models with constant domains. Then* T^{c} *is conservative over* T^{i} *with respect to the class of prenex formulae with a positive formula as the matrix.*

PROOF. The proof goes along the lines of that of Theorem 5.6 but we refer to the second part of Lemma 5.5. ■

Let us note that in Theorem 5.6 we do not assume that the theory in question is closed under the negative translation. Instead, we need a semantic property that the theory is complete with respect to a class of conversely well-founded Kripke models.

6. Applications

In this section we provide applications of the results presented in this paper to Constructive Zermelo Fraenkel set theory CZF. For the background, axiomatization and properties of CZF see [8,10].

In general, constructing Kripke models for a particular intuitionistic theory is a difficult task. In case of fragments of CZF, some constructions were introduced in [8]. The results presented there involve conversely well-founded and normal models.

The key result which enables us to describe models of CZF^- (which is CZF without Collection Axioms) can be rephrased in the following way (cf. [8, Corollary 9]).

Let $\mathcal{M} = (W, \leq, \{\mathbf{M}_w : w \in W\})$ *be a Kripke model such that*

(a) *the frame* $(W, \leq)$ *of* $\mathcal{M}$ *has no infinite branches,*

(b) *all the worlds* $\mathbf{M}_w$ *of* $\mathcal{M}$ *are transitive models of* ZF*,*
(c) *all atomic formulae are decidable in* $\mathcal{M}$*.*

Then $\mathcal{M}$ *is a model of* CZF^- *plus Bounded Strong Collection and Set-bounded Subset Collection.*

The result above allows us to define a very natural class of models of some extension of CZF^-. Let us denote by $\mathbb{F}$ the class of all Kripke models that satisfy the above conditions. We define $\mathsf{CZF}^{\mathbb{F}}$ as the theory of the class $\mathbb{F}$, in other words,

$$\mathsf{CZF}^{\mathbb{F}} := \bigcap\{\mathrm{Th}(\mathcal{M}) \ : \ \mathcal{M} \in \mathbb{F}\}.$$

Of course, the theory $\mathsf{CZF}^{\mathbb{F}}$ contains CZF^- and proves Bounded of Strong Collection and Set-bounded Subset Collection. Moreover, any path in every model of $\mathbb{F}$ is finite, therefore $\mathsf{CZF}^{\mathbb{F}}$ contains also formulae which are not intuitionistically valid, for example the Principle of Double Negation Shift (DNS), $\forall x \neg\neg A \to \neg\neg\forall x A$.

Since the frames of the models of the class $\mathbb{F}$ have no infinite branches, and all the worlds of the Kripke models $\mathcal{M}$ of $\mathbb{F}$ are models of ZF, we have $\mathcal{M} \Vdash \neg\neg A$, for all $\mathcal{M} \in \mathbb{F}$ and A such that $\mathsf{ZF} \vdash A$. It follows that every theorem of ZF prefixed by double negation belongs to $\mathsf{CZF}^{\mathbb{F}}$, and hence

$$(\mathsf{CZF}^{\mathbb{F}})^c = \mathsf{ZF}.$$

Obviously, $\mathsf{CZF}^{\mathbb{F}}$ is complete with the class of ZF-normal models, so it satisfies the assumptions of Theorem 4.10. Hence we derive the following fact.

COROLLARY 6.1. ZF *is conservative over* $\mathsf{CZF}^{\mathbb{F}}$ *with respect to the formulae of the form*

$$\forall \bar{x}(C \to \forall \bar{y} \exists \bar{z} D),$$

where C *is a generalized semi-positive formula in* CZF^f *and formula* D *is quantifier-free.*

Note also that $\mathsf{CZF}^{\mathbb{F}}$ is complete with respect to a class of conversely well-founded Kripke models and it is closed under the Friedman translation, so it satisfies also the assumptions of Theorems 5.6.

Finally, observe that $\mathsf{CZF}^{\mathbb{F}}$ depends on axioms which essentially involve disjunctions and existential quantifiers and is not closed under the negative translation. Thus, Corollary 6.1 cannot be proven by the syntactic methods used in the proof of Theorem 2.1.

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