Title: On a problem of Janusz Matkowski and Jacek Wesołowski, II

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Citation style: Morawiec Janusz, Zürcher Thomas. (2019). On a problem of Janusz Matkowski and Jacek Wesołowski, II. "Aequationes mathematicae" (Vol. 93, iss. 1 (2019), s. 91-108), doi 10.1007/s00010-018-00636-3

Aequat. Math. 93 (2019), 91–108

0001-9054/19/010091-18
published online February 8, 2019
https://doi.org/10.1007/s00010-018-00636-3

Aequationes Mathematicae

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On a problem of Janusz Matkowski and Jacek Wesołowski, II

Janusz Morawiec and Thomas Zürcher

Dedicated to Professor Karol Baron on the occasion of his 70th birthday.

Abstract. We continue our study started in Morawiec and Zürcher (Aequ Math 92(4):601–615, 2018) of the functional equation

$$\varphi(x) = \sum_{n=0}^{N} \varphi(f_n(x)) - \sum_{n=0}^{N} \varphi(f_n(0))$$

and its increasing and continuous solutions $\varphi\colon [0,1] \to [0,1]$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$. In this paper we assume that $f_0, \dots, f_N\colon [0,1] \to [0,1]$ are strictly increasing contractions such that

$$0 \le f_0(0) < f_0(1) \le f_1(0) < \cdots < f_{N-1}(1) \le f_N(0) < f_N(1) \le 1$$

and at least one of the weak inequalities is strong.

Mathematics Subject Classification. Primary 39B12, Secondary 37A05.

Keywords. Functional equations, Iterated function systems, Singular functions, Absolutely continuous functions.

1. Introduction

Fix $N \in \mathbb{N}$ and strictly increasing contractions $f_0, \dots, f_N\colon [0,1] \to [0,1]$ such that

$$0 \le f_0(0) < f_0(1) \le f_1(0) < \cdots < f_{N-1}(1) \le f_N(0) < f_N(1) \le 1. \qquad (1)$$

We continue our study of the existence of solutions φ of the functional equation

$$\varphi(x) = \sum_{n=0}^{N} \varphi(f_n(x)) - \sum_{n=0}^{N} \varphi(f_n(0)) \qquad \text{(E)}$$

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in the class $\mathcal{C}$ consisting of all increasing and continuous functions $\varphi\colon [0,1] \to [0,1]$ satisfying the following boundary conditions

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi(1) = 1. \tag{2}$$

In this paper we assume, in contrast to [11], that

$$\bigcup_{n=0}^{N} \big[f_n(0), f_n(1)\big] \neq [0,1]. \tag{3}$$

2. Preliminaries

Throughout this paper for all $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \{0, \ldots, N\}$ we denote the composition $f_{n_1} \circ \cdots \circ f_{n_k}$ by $f_{n_1,\ldots,n_k}$. Moreover, we extend the notation to the case $k = 0$ by letting $f_{n_1,\ldots,n_0}$ be the identity.

We begin with three lemmas. The proof of the first one is very easy, so we omit it.

Lemma 2.1. *Fix $m \in \mathbb{N}$ and nonnegative real numbers $\alpha_1, \ldots, \alpha_m$ such that $\sum_{i=1}^m \alpha_i = 1$. If $\varphi_1, \ldots, \varphi_m \in \mathcal{C}$, then $\sum_{i=1}^m \alpha_i \varphi_i \in \mathcal{C}$.*

Lemma 2.2. *If $\varphi \in \mathcal{C}$, then $\varphi(f_0(0)) = 0$ and $\varphi(f_N(1)) = 1$.*

Proof. By (2), (E), (1), and the monotonicity of φ we have

$$\begin{aligned} 1 = \varphi(1) &= \varphi(f_N(1)) - \varphi(f_0(0)) + \sum_{n=1}^{N} \big[\varphi(f_{n-1}(1)) - \varphi(f_n(0))\big] \\ &\le \varphi(f_N(1)) - \varphi(f_0(0)). \end{aligned}$$

As the image of $[0,1]$ under φ lies in $[0,1]$, we infer that $\varphi(f_0(0)) = 0$ and $\varphi(f_N(1)) = 1$. □

Now we want to show that if all the contractions $f_0, \ldots, f_N$ are *nonsingular* (i.e. $f_0^{-1}(A), \ldots, f_N^{-1}(A)$ have Lebesgue measure zero for every set $A \subset [0,1]$ of Lebesgue measure zero,[1]) then the class $\mathcal{C}$ is determined by two of its subclasses $\mathcal{C}_a$ and $\mathcal{C}_s$ of all absolutely continuous and all singular functions, respectively. Repeating directly the proof of Remark 2.2 from [11] with the use of Lemma 2.2 we get the following result.

Lemma 2.3. *Assume that all the contractions $f_0, \ldots, f_N$ are nonsingular. Then, both the absolutely continuous and the singular parts[2] of every element from $\mathcal{C}$ satisfy (E) for every $x \in [0,1]$.*

[1] See [7]. Note also that as the inverses of the contractions exist and are continuous and increasing, being nonsingular is equivalent to the inverses being absolutely continuous, see for example Theorem 7.1.38 in [6].

[2] The parts are unique up to a constant. For definiteness, we choose them so that both of them map 0 to 0.

By the monotonicity of f_0 and f_N, it is easy to prove that the sequence $(f_{\underbrace{0,\dots,0}_{k}}(0))_{k\in\mathbb{N}}$ is increasing and the sequence $(f_{\underbrace{N,\dots,N}_{k}}(1))_{k\in\mathbb{N}}$ is decreasing. Hence both are convergent. Put

$$\mathbf{0}=\lim_{k\to\infty} f_{\underbrace{0,\dots,0}_{k}}(0) \quad \text{and} \quad \mathbf{1}=\lim_{k\to\infty} f_{\underbrace{N,\dots,N}_{k}}(1).$$

It is clear that $\mathbf{0}$ is the unique fixed point of f_0 and $\mathbf{1}$ is the unique fixed point of f_N, i.e.

$$f_0(\mathbf{0})=\mathbf{0} \quad \text{and} \quad f_N(\mathbf{1})=\mathbf{1}. \tag{4}$$

Moreover,

$$\mathbf{1}=\lim_{k\to\infty} f_{\underbrace{N,\dots,N}_{k}}(0),$$

because for every $k\in\mathbb{N}$ we have $|f_{\underbrace{N,\dots,N}_{k}}(1)-f_{\underbrace{N,\dots,N}_{k}}(0)|\le c^k$, where $c\in(0,1)$ is a Lipschitz constant of f_N.

Lemma 2.4. *Assume that $\varphi\in\mathcal{C}$. Then $\varphi(\mathbf{0})=0$ and $\varphi(\mathbf{1})=1$.*

Proof. We first prove that $\varphi(\mathbf{0})=0$.

By Lemma 2.2 we have $\varphi(f_0(0))=0$. Fix $k\in\mathbb{N}$ and assume inductively that $\varphi(f_{\underbrace{0,\dots,0}_{k}}(0))=0$. Applying the induction hypothesis, (E), Lemma 2.2 and the monotonicity of $f_0,\dots,f_N$ and φ, we get

$$\begin{aligned}
0=\varphi(f_{\underbrace{0,\dots,0}_{k}}(0)) &= \sum_{n=0}^{N}\varphi(f_n(f_{\underbrace{0,\dots,0}_{k}}(0)))-\sum_{n=0}^{N}\varphi(f_n(0))\\
&=\varphi(f_{\underbrace{0,\dots,0}_{k+1}}(0))+\sum_{n=1}^{N}\varphi(f_n(f_{\underbrace{0,\dots,0}_{k}}(0)))-\sum_{n=1}^{N}\varphi(f_n(0))\\
&\ge\varphi(f_{\underbrace{0,\dots,0}_{k+1}}(0))\ge 0.
\end{aligned}$$

Hence $\varphi(f_{\underbrace{0,\dots,0}_{k+1}}(0))=0$. Now the continuity of φ gives

$$\varphi(\mathbf{0})=\lim_{k\to\infty}\varphi(f_{\underbrace{0,\dots,0}_{k}}(0))=0.$$

To prove that $\varphi(\mathbf{1})=1$ observe first that by (1) and the monotonicity of φ we have $\varphi(f_n(1))\le\varphi(f_{n+1}(0))$ for every $n\in\{0,\dots,N-1\}$. We want to show that

$$\varphi(f_n(1))=\varphi(f_{n+1}(0)) \tag{5}$$

for every $n \in \{0, \dots, N-1\}$. Suppose that, contrary to our claim, there exists $n \in \{0, \dots, N-1\}$ such that $\varphi(f_n(1)) < \varphi(f_{n+1}(0))$. Then, using Lemma 2.2 and arguing as in its proof, we obtain

$$\begin{aligned}1 = \varphi(1) &= \sum_{n=0}^{N-1} \varphi(f_n(1)) + 1 - \sum_{n=0}^{N} \varphi(f_n(0)) \\ &< \sum_{n=0}^{N-1} \varphi(f_{n+1}(0)) + 1 - \sum_{n=0}^{N} \varphi(f_n(0)) = 1 - \varphi(f_0(0)) = 1,\end{aligned}$$

a contradiction.

Now we show by induction that

$$\varphi(f_{\underbrace{N,\dots,N}_{k}}(1)) = 1 \tag{6}$$

for all $k \in \mathbb{N}$. The first step of the induction holds due to Lemma 2.2. Fix $k \in \mathbb{N}$ and assume that (6) holds. Then applying (6), (E), Lemma 2.2, (5) and the monotonicity of $f_0, \dots, f_N$ and φ we get

$$\begin{aligned}1 = \varphi(f_{\underbrace{N,\dots,N}_{k}}(1)) &= \sum_{n=0}^{N} \varphi(f_n(f_{\underbrace{N,\dots,N}_{k}}(1))) - \sum_{n=1}^{N} \varphi(f_n(0)) \\ &\le \sum_{n=0}^{N-1} \varphi(f_n(f_{\underbrace{N,\dots,N}_{k}}(1))) + \varphi(f_{\underbrace{N,\dots,N}_{k+1}}(1)) - \sum_{n=0}^{N-1} \varphi(f_n(1)) \\ &\le \varphi(f_{\underbrace{N,\dots,N}_{k+1}}(1)) \le 1.\end{aligned}$$

Hence $\varphi(f_{\underbrace{N,\dots,N}_{k+1}}(1)) = 1$. Finally, passing with k to infinity in (6) and using the continuity of φ we obtain $\varphi(\mathbf{1}) = 1$. $\square$

3. Basic property of solutions

Define recursively a sequence $(A_k)_{k\in\mathbb{N}}$ of subsets of the interval $[0,1]$ as follows:

$$A_0 = [0,1] \quad \text{and} \quad A_k = \bigcup_{n=0}^{N} f_n(A_{k-1}) \quad \text{for every } k \in \mathbb{N}.$$

By (3) we have $A_1 = \bigcup_{n=0}^{N} \left[f_n(0), f_n(1)\right] \subsetneq A_0$. Moreover, a witness of the strict inclusion can be found that is different from 0 and 1. This jointly with an easy induction shows that $A_{k+1} \subsetneq A_k$ for every $k \in \mathbb{N}$. Again there is a witness of the strict inequality differing from 0 and 1. Put

$$A_* = \bigcap_{k\in\mathbb{N}} A_k.$$

It is clear that A_* is compact and

$$A_* = \bigcup_{n=0}^{N} f_n(A_*). \tag{7}$$

We will show that the just constructed set A_*, called the *attractor* of the iterated function system $\{f_0, \ldots, f_N\}$ (see [1]), is a *Cantor-like set*, i.e. uncountable, nowhere dense and a perfect subset of $\mathbb{R}$ (see [13]); note that A_* is uncountable and nowhere dense, which follows from its construction. Moreover, we will see in Theorem 3.6 that A_* is perfect and in Example 3.5 that it is of Lebesgue measure zero if $f_0, \ldots, f_N$ are similitudes, whereas in the general case it can happen that A_* is of positive Lebesgue measure (see [10]).

From the construction we have

$$A_* = \bigcap_{k\in\mathbb{N}} \left(\bigcup_{n_1,\ldots,n_k\in\{0,\ldots,N\}} \left[f_{n_1,\ldots,n_k}(0), f_{n_1,\ldots,n_k}(1)\right] \right).$$

Whenever a point x can be written as

$$x = \lim_{k\to\infty} f_{x_1,\ldots,x_k}(0) = \lim_{k\to\infty} f_{x_1,\ldots,x_k}(1), \tag{8}$$

we say that x has an *address*[3] (see [1]).

Lemma 3.1. *The set A_* is exactly the set of points in $[0,1]$ that have an address.*

Proof. Let $x \in A_*$. Note that for every $k \in \mathbb{N}$ there exist $x_1^k, \ldots, x_k^k \in \{0, \ldots, N\}$ such that $x \in [f_{x_1^k,\ldots,x_k^k}(0), f_{x_1^k,\ldots,x_k^k}(1)]$ with x_m^n not necessarily agreeing with x_m^l for different l and n, however, as each x_m^l is chosen from the finite set $\{0, \ldots, N\}$, we may apply a Cantor diagonal argument to get a sequence as wished.

It is easy to see that every sequence $(x_k)_{k\in\mathbb{N}}$ of elements of $\{0, \ldots, N\}$ is an address of a point from the set A_*. □

Note that

$$\mathbf{0} = \min A_* \quad \text{and} \quad \mathbf{1} = \max A_*. \tag{9}$$

Since A_* is a closed set, it follows that $[\mathbf{0}, \mathbf{1}]\backslash A_*$ is an open set. Moreover,

$$[\mathbf{0}, \mathbf{1}]\backslash A_* = \bigcup_{k\in\mathbb{N}} \bigcup_{\substack{0\le n_1,\ldots,n_{k-1}\le N \\ 0\le n_k\le N-1}} \left(f_{n_1,\ldots,n_k}(\mathbf{1}), f_{n_1,\ldots,n_k+1}(\mathbf{0})\right) \tag{10}$$

[3] We have come across the term *coding* as well.

and for all $k \in \mathbb{N}$, $n_1, \dots, n_{k-1} \in \{0, \dots, N\}$ and $n_k \in \{0, \dots, N-1\}$ the interval $(f_{n_1,\dots,n_k}(\mathbf{1}), f_{n_1,\dots,n_k+1}(\mathbf{0}))$ is a connected component of the set $[\mathbf{0},\mathbf{1}]\setminus A_*$.

Now we are in a position to show that any $\varphi \in \mathcal{C}$ is constant on the closure of each connected component of the set $[0,1]\setminus A_*$. We do it in two steps.

Lemma 3.2. *Assume that* $\varphi \in \mathcal{C}$. *Then:*

(i) $\varphi|_{[0,\mathbf{0}]} = 0$;
(ii) $\varphi|_{[\mathbf{1},1]} = 1$;
(iii) $\varphi|_{[f_n(\mathbf{1}), f_{n+1}(\mathbf{0})]}$ *is constant for every* $n \in \{0, \dots, N-1\}$.

Proof. To prove (i) and (ii) it is enough to apply Lemma 2.4 jointly with the monotonicity of φ.

Let us tackle (iii). According to (5) and to the monotonicity of $f_0, \dots, f_N$ and φ, we see that $\varphi(f_n(\mathbf{1})) \le \varphi(f_{n+1}(\mathbf{0}))$. Suppose that, contrary to our claim, there exists $n \in \{0, \dots, N-1\}$ such that $\varphi(f_n(\mathbf{1})) < \varphi(f_{n+1}(\mathbf{0}))$. Then, using Lemma 2.4, (E), and the first equality of (4) we get

$$\begin{aligned} 1 = \varphi(\mathbf{1}) &= \sum_{n=0}^{N-1} \varphi(f_n(\mathbf{1})) + \varphi(f_N(\mathbf{1})) - \sum_{n=0}^{N} \varphi(f_n(0)) \\ &< \sum_{n=0}^{N-1} \varphi(f_{n+1}(\mathbf{0})) + 1 - \sum_{n=0}^{N} \varphi(f_n(0)) = \varphi(\mathbf{0}) - \varphi(f_0(\mathbf{0})) + 1 = 1, \end{aligned}$$

a contradiction. □

Lemma 3.3. *Assume that* $\varphi \in \mathcal{C}$. *Then for all* $k \in \mathbb{N}$, $n_1, \dots, n_{k-1} \in \{0, \dots, N\}$ *and* $n_k \in \{0, \dots, N-1\}$ *there exists* $c_{n_1,\dots,n_k} \in [0,1]$ *such that*

$$\varphi|_{[f_{n_1,\dots,n_k}(\mathbf{1}), f_{n_1,\dots,n_k+1}(\mathbf{0})]} = c_{n_1,\dots,n_k}. \tag{11}$$

Proof. We proceed by induction on k.

The first step of the induction is implied by assertion (iii) of Lemma 3.2.

Fix $k \in \mathbb{N}$, $n_1, \dots, n_{k-1} \in \{0, \dots, N\}$, $n_k \in \{0, \dots, N-1\}$ and assume that there exists $c_{n_1,\dots,n_k} \in [0,1]$ such that (11) holds. Then (11), (E) and the monotonicity of $f_0, \dots, f_N$ and φ imply

$$\begin{aligned} c_{n_1,\dots,n_k} = \varphi(f_{n_1,\dots,n_k}(\mathbf{1})) &= \sum_{n=0}^{N} \varphi(f_{n,n_1,\dots,n_k}(\mathbf{1})) - \sum_{n=0}^{N} \varphi(f_n(0)) \\ &\le \sum_{n=0}^{N} \varphi(f_{n,n_1,\dots,n_k+1}(\mathbf{0})) - \sum_{n=0}^{N} \varphi(f_n(0)) = \varphi(f_{n_1,\dots,n_k+1}(\mathbf{0})) \\ &= c_{n_1,\dots,n_k}. \end{aligned}$$

Hence

$$\sum_{n=0}^{N} \varphi(f_{n,n_1,\dots,n_k}(\mathbf{1})) = \sum_{n=0}^{N} \varphi(f_{n,n_1,\dots,n_k+1}(\mathbf{0})),$$

and applying again the monotonicity of $f_0, \dots, f_N$ and φ, we obtain

$$\varphi(f_{n,n_1,\dots,n_k}(\mathbf{1})) = \varphi(f_{n,n_1,\dots,n_k+1}(\mathbf{0}))$$

for every $n \in \{0, \dots, N\}$. □

Combining Lemmas 3.2 and 3.3 with (10), we get the following result.

Theorem 3.4. *If the set A_* has Lebesgue measure zero, then $\mathcal{C} = \mathcal{C}_s$.*

We now give an example of contractions $f_0, \dots, f_N$ for which the set A_* is of Lebesgue measure zero.

Example 3.5. Assume additionally to our assumptions in the introduction that $f_0, \dots, f_N$ are similitudes, i.e.

$$f_n(x) = (\beta_n - \alpha_n)x + \alpha_n$$

for all $x \in [0, 1]$ and $n \in \{0, \dots, N\}$, where

$$0 \le \alpha_0 < \beta_0 \le \alpha_1 < \beta_1 \le \cdots \le \alpha_N < \beta_N \le 1 \quad \text{and} \quad \bigcup_{n=0}^{N} [\alpha_n, \beta_n] \ne [0, 1].$$

Clearly, (1) and (3) hold. Denote by l the Lebesgue measure on the real line and put $d = l(A_0 \backslash A_1)$. By a simple induction we get $l(A_k \backslash A_{k+1}) = d(1 - d)^k$ for every $k \in \mathbb{N}$. From (1) and (3) we infer that $d \in (0, 1)$ and hence that

$$l(A_*) = 1 - \sum_{k=0}^{\infty} l(A_k \backslash A_{k+1}) = 1 - \frac{d}{1 - (1 - d)} = 0.$$

We finish this section with one more property of the set A_*.

Theorem 3.6. *The set A_* is perfect.*

Proof. We know from its definition that A_* is closed, and it is nonempty by (9).

Let $x \in A_*$ and fix an address of x, i.e. a sequence $(x_k)_{k\in\mathbb{N}}$ of elements of $\{0, \dots, N\}$ satisfying (8); we can choose such a sequence according to Lemma 3.1. To complete the proof, we need to show that in each neighbourhood of x we can find some element belonging to $A_* \backslash \{x\}$.

Fix $\varepsilon > 0$ and $m \in \mathbb{N}$ so large that $L^{m-1} < \varepsilon$, where $L \in (0, 1)$ is the largest Lipschitz constant of the given contractions $f_0, \dots, f_N$. Define a sequence $(y_k)_{k\in\mathbb{N}}$ by putting $y_k = x_k$ for all $k \ne m$ and choosing arbitrarily $y_m \in \{0, \dots, N\} \backslash \{x_m\}$. Then

$$y = \lim_{k\to\infty} f_{y_1,\dots,y_k}(0) \in A_*.$$

Since all considered contractions are injective and the addresses of points x and y differ only in the m-th coordinate, it follows that $y \ne x$. Moreover,

$$\begin{aligned} |x - y| &= \lim_{k\to\infty} |f_{x_1,\dots,x_{m-1}}(f_{x_m,\dots,x_k}(0)) - f_{x_1,\dots,x_{m-1}}(f_{y_m,\dots,y_k}(0))| \\ &\le L^{m-1} \lim_{k\to\infty} |f_{x_m,\dots,x_k}(0) - f_{y_m,\dots,y_k}(0)| \le L^{m-1} < \varepsilon. \end{aligned}$$

The proof is complete. □

4. Existence of solutions

In the previous section we have discussed the behaviour of functions belonging to the class $\mathcal{C}$, but, up to now, we do not know if $\mathcal{C}$ contains any function at all. In this section, we want to show that $\mathcal{C} \neq \emptyset$.

Fix positive real numbers $p_0, \ldots, p_N$ such that

$$\sum_{n=0}^{N} p_n = 1. \tag{12}$$

Then there exists a unique Borel probability measure μ such that

$$\mu(B) = \sum_{n=0}^{N} p_n \mu(f_n^{-1}(B)) \tag{13}$$

for every Borel set $B \subset [0,1]$ (see [5]; cf. [4]). From now on the letter μ will be reserved for the unique Borel probability measure satisfying (13) for every Borel set $B \subset [0,1]$.

Now we are interested in some properties of the measure μ that will be needed later. We begin with a well-known folklore lemma; for its proof the reader can consult [8].

Lemma 4.1. *The measure μ is either singular or absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

To formulate the next lemma, which is also well-known (see e.g. [1]), we recall that the support of the measure μ is the set $\operatorname{supp}\mu$ of all points $x \in [0,1]$ such that $\mu([x-\varepsilon, x+\varepsilon]) > 0$ for every $\varepsilon > 0$.

Lemma 4.2. *We have* $\operatorname{supp}\mu = A_*$. *In particular,* $\mu([0,1] \setminus A_*) = 0$.

Lemma 4.3. *The measure μ is continuous.*

Proof. To prove that μ is continuous it is enough to show $\mu(\{x\}) = 0$ for every $x \in [0,1]$.

Fix $x \in [0,1]$.

If $x \notin A_*$, then $\mu(\{x\}) = 0$ by Lemma 4.2, hence we assume now that $x \in A_*$ and choose an address of x, that is a sequence $(x_k)_{k\in\mathbb{N}}$ of elements of $\{0,\ldots,N\}$ such that (8) holds. Note that the monotonicity of $f_0,\ldots,f_N$ implies

$$f_{x_1,\ldots,x_k}(0) \le f_{x_1,\ldots,x_{k+1}}(0) \le x \le f_{x_1,\ldots,x_{k+1}}(1) \le f_{x_1,\ldots,x_k}(1) \tag{14}$$

for every $k \in \mathbb{N}$.

First we want to prove that

$$\mu\big(\{f_n(0)\}\big) = \mu\big(\{f_n(1)\}\big) = 0 \tag{15}$$

for every $n \in \{0, \dots, N\}$.

We begin with proving that

$$\mu(\{0\}) = \mu(\{1\}) = 0. \tag{16}$$

If $f_0(0) > 0$, then (13) gives

$$\mu(\{0\}) = \sum_{n=0}^{N} p_n \mu(\{f_n^{-1}(0)\}) = \sum_{n=0}^{N} p_n \mu(\emptyset) = 0.$$

If $f_0(0) = 0$, then (13) yields

$$\mu(\{0\}) = \sum_{n=0}^{N} p_n \mu(\{f_n^{-1}(0)\}) = p_0 \mu(\{0\}) + \sum_{n=1}^{N} p_n \mu(\emptyset) = p_0 \mu(\{0\}),$$

and since $p_0 \in (0,1)$ we conclude that $\mu(\{0\}) = 0$.

In the same way, considering two cases ($f_N(1) < 1$ and $f_N(1) = 1$) and using (13) jointly with the fact that $p_N \in (0,1)$ in the second case, we get $\mu(\{1\}) = 0$.

Using (13), (1) and (16) we obtain

$$\mu(\{f_0(0)\}) = \sum_{n=0}^{N} p_n \mu(\{f_n^{-1}(f_0(0))\}) = p_0 \mu(\{0\}) = 0,$$

$$\mu(\{f_N(1)\}) = \sum_{n=0}^{N} p_n \mu(\{f_n^{-1}(f_N(1))\}) = p_N \mu(\{1\}) = 0$$

and

$$\begin{aligned}\mu(\{f_m(1), f_{m+1}(0)\}) &= \sum_{n=0}^{N} p_n \mu(\{f_n^{-1}(f_m(1)), f_n^{-1}(f_{m+1}(0))\}) \\ &\leq 2(p_m \mu(\{1\}) + p_{m+1} \mu(\{0\})) = 0\end{aligned}$$

for every $m \in \{0, \dots, N-1\}$.

By (15) and (1), equality (13) implies

$$\mu(f_n(B)) = p_n \mu(B) \tag{17}$$

for all $n \in \{0, \dots, N\}$ and Borel sets $B \subset [0,1]$.

Finally, applying (14) and equality (17) k times jointly with the fact that μ is a probability measure we get

$$\begin{aligned}\mu(\{x\}) &= \mu\left(\bigcap_{k\in\mathbb{N}}\left[f_{x_1,\ldots,x_k}(0), f_{x_1,\ldots,x_k}(1)\right]\right)\\ &= \lim_{k\to\infty}\mu\left(\left[f_{x_1,\ldots,x_k}(0), f_{x_1,\ldots,x_k}(1)\right]\right)\\ &= \lim_{k\to\infty}\prod_{i=1}^{k}p_{x_i} \le \lim_{k\to\infty}\left(\max\{p_0,\ldots,p_N\}\right)^k = 0.\end{aligned}$$

The proof is complete. □

Combining Lemmas 4.2 and 4.3 with (10) we get the following corollary.

Corollary 4.4. *The measure μ vanishes on each of the intervals:* $[0,\mathbf{0}]$, $[\mathbf{1},1]$, $[f_{n_1,\ldots,n_k}(\mathbf{1}), f_{n_1,\ldots,n_k+1}(\mathbf{0})]$ *with* $k\in\mathbb{N}$, $n_1,\ldots,n_{k-1}\in\{0,\ldots,N\}$ *and* $n_k\in\{0,\ldots,N-1\}$.

Define the function $\varphi\colon[0,1]\to[0,1]$ by

$$\varphi(x) = \mu([0,x]).$$

From now on the letter φ will be reserved for the just defined function.

Repeating the proof of Theorem 3.3 from [11] we get the following result.

Theorem 4.5. *Either* $\varphi\in\mathcal{C}_a$ *or* $\varphi\in\mathcal{C}_s$.

As a consequence of Theorem 4.5, we have $\varphi\in\mathcal{C}$. Lemma 4.2 implies that φ cannot be constant on an open interval having nonempty intersection with the attractor A_*. Therefore, all the constants $c_{n_1,\ldots,n_k}$ occurring in the assertion of Lemma 3.3 (associated with the above constructed φ) are pairwise different and belong to the open interval $(0,1)$.

We finish this section by giving a precise formula for φ.

Theorem 4.6. *Assume that* $x\in[0,1]$.

(i) *If* $x\in[0,\mathbf{0}]$, *then* $\varphi(x)=0$.
(ii) *If* $x\in[\mathbf{1},1]$, *then* $\varphi(x)=1$.
(iii) *If* $x\in A_*$ *and* $(x_l)_{l\in\mathbb{N}}$ *is an address of* x, *then*

$$\varphi(x) = \sum_{l=1}^{\infty}\operatorname{sgn}(x_l)\left[\prod_{n=0}^{N}p_n^{\#\{i\in\{1,\ldots,l-1\}:x_i=n\}}\cdot\sum_{n=0}^{x_l-1}p_n\right].$$

(iv) *If* $x \in [f_{x_1,\dots,x_k}(\mathbf{1}), f_{x_1,\dots,x_k+1}(\mathbf{0})]$ *with* $k \in \mathbb{N}$, $x_1,\dots,x_{k-1} \in \{0,\dots,N\}$ *and* $x_k \in \{0,\dots,N-1\}$, *then*

$$\varphi(x) = \sum_{l=1}^{k} \operatorname{sgn}(x_l) \left[\prod_{n=0}^{N} p_n^{\#\{i\in\{1,\dots,l-1\}:x_i=n\}} \cdot \sum_{n=0}^{x_l-1} p_n \right] + \prod_{n=0}^{N} p_n^{\#\{i\in\{1,\dots,k\}:x_i=n\}}.$$

Proof. Assertions (i) and (ii) are trivially implied by assertions (i) and (ii) of Lemma 3.2. The proof of assertion (iii) follows very closely the proof of Theorem 3.6 from [11], so we omit it. Assertion (iv) is a consequence of Lemma 3.3, the fact that $(x_1,\dots,x_k,N,\dots)$ is the address of the point $f_{x_1,\dots,x_k}(\mathbf{1})$ and assertion (iii); indeed

$$\begin{aligned}\varphi(x) = \varphi(f_{x_1,\dots,x_k}(\mathbf{1})) &= \sum_{l=1}^{k} \operatorname{sgn}(x_l) \left[\prod_{n=0}^{N} p_n^{\#\{i\in\{1,\dots,l-1\}:x_i=n\}} \cdot \sum_{n=0}^{x_l-1} p_n \right] \\ &\quad + \sum_{l=k+1}^{\infty} \left[\prod_{n=0}^{N} p_n^{\#\{i\in\{1,\dots,k\}:x_i=n\}} p_N^{l-k-1} \right] \cdot \big(1-p_N\big) \\ &= \sum_{l=1}^{k} \operatorname{sgn}(x_l) \left[\prod_{n=0}^{N} p_n^{\#\{i\in\{1,\dots,l-1\}:x_i=n\}} \cdot \sum_{n=0}^{x_l-1} p_n \right] \\ &\quad + \prod_{n=0}^{N} p_n^{\#\{i\in\{1,\dots,k\}:x_i=n\}}.\end{aligned}$$

This finishes the proof. □

5. More about the class $\mathcal{C}$

As we have seen in Theorem 4.5, with each sequence $\big(p_0,\dots,p_N\big)$ of positive real numbers satisfying (12) we have associated a continuous increasing surjective solution $\varphi_{p_0,\dots,p_N} : [0,1] \to [0,1]$ of Eq. (E). We denote by $\mathcal{W}$ the set of all these solutions. The main purpose of this section is to prove the following result.

Theorem 5.1. *The set* $\mathcal{W}$ *is linearly independent and its convex hull is contained in* $\mathcal{C}$.

5.1. Proof of Theorem 5.1

The statement concerning the convex hull follows from Lemma 2.1.

The proof of the independence will be divided into several lemmas. Before we formulate the first one, note that for every $y \in A_*$ equality (7) guarantees that there exists at least one $n \in \{0, \dots, N\}$ such that $y \in f_n(A_*)$. Therefore, we can define a transformation $T\colon A_* \to [0,1]$ by putting

$$T(y) = f_{n(y)}^{-1}(y),$$

where

$$n(y) = \max\{n \in \{0, \dots, N\} : y \in f_n(A_*)\}.$$

Lemma 5.2. *The transformation T maps A_* into A_* and it is measure preserving for μ.*

Proof. To see that $T(A_*) \subset A_*$ we fix $y \in A_*$. Then the injectivity of $f_{n(y)}$ implies that there is exactly one $x \in A_*$ such that $y = f_{n(y)}(x)$. Thus $T(y) = x \in A_*$.

Now we prove that T is measure preserving for μ.

Fix a Borel set $B \subset A_*$. As $A_* \subset \bigcup_{n=0}^{N} [f_n(0), f_n(1)]$, we have

$$T^{-1}(B) = \bigcup_{n=0}^{N} \{y \in [f_n(0), f_n(1)] \cap A_* : T(y) \in B\}.$$

Then using Lemma 4.3 jointly with (1) and the fact that the set $\bigcup_{i=0}^{N} f_i^{-1}(y)$ contains just one element in the case where $y \in (f_n(0), f_n(1))$ and at most two elements in the case where $y \in \{f_n(0), f_n(1)\}$, we obtain

$$\begin{aligned} \mu(T^{-1}(B)) &= \sum_{n=0}^{N} \mu(\{y \in [f_n(0), f_n(1)] \cap A_* : T(y) \in B\}) \\ &= \sum_{n=0}^{N} \mu(\{y \in [f_n(0), f_n(1)] \cap A_* : f_n^{-1}(y) \in B\}) \\ &= \sum_{n=0}^{N} \mu(\{y \in [f_n(0), f_n(1)] \cap A_* : y \in f_n(B)\}). \end{aligned}$$

Next note that $f_n(B) \subset f_n(A_*) \subset A_*$ for every $n \in \{0, \dots, N\}$. Thus

$$\mu(T^{-1}(B)) = \sum_{n=0}^{N} \mu(f_n(B)).$$

Finally, according to (17) we conclude that

$$\mu(T^{-1}(B)) = \sum_{n=0}^{N} p_n \mu(B) = \mu(B),$$

and the proof is complete. □

By Lemma 3.1 the points in A_* are exactly the ones that have an address. The next lemma shows that we might run into slight problems with the uniqueness of the addresses if

$$f_0(0) = 0, \quad f_N(1) = 1 \quad \text{and} \quad N_b \neq \emptyset, \tag{18}$$

where

$$N_b = \{n \in \{0, \ldots, N-1\} : f_n(1) = f_{n+1}(0)\}.$$

Lemma 5.3.

(i) *Every point from A_* has at most two addresses, and if a point from A_* has two addresses, then (18) holds and exactly one of the addresses belongs to the set*

$$Z_b = \left\{(x_k)_{k\in\mathbb{N}} \in \{0, \ldots, N\}^{\mathbb{N}} : \exists_{n\in\mathbb{N}}(x_n \in N_b \text{ and } x_k = N \text{ for every } k > n)\right\}.$$

(ii) *If (18) holds and a point from A_* has an address belonging to the set Z_b, then it also has an address not belonging to the set Z_b.*

(iii) *Every point from A_* has exactly one address if and only if (18) does not hold.*

Proof. (i) Assume that $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ are two different addresses of a point $x \in A_*$. Put

$$m = \min\{k \in \mathbb{N} : x_k \neq y_k\}$$

and let $x_m < y_m$. Then according to (8) and (1) we have

$$\begin{aligned} x &= f_{x_1,\ldots,x_m}\left(\lim_{k\to\infty} f_{x_{m+1},\ldots,x_k}(1)\right) \\ &\leq f_{x_1,\ldots,x_m}(1) \leq f_{y_1,\ldots,y_m}(0) \leq f_{y_1,\ldots,y_m}\left(\lim_{k\to\infty} f_{y_{m+1},\ldots,y_k}(0)\right) = x. \end{aligned}$$

Thus $f_{x_1,\ldots,x_m}(1) = f_{y_1,\ldots,y_m}(0)$, and hence $f_{x_m}(1) = f_{y_m}(0) \in A_*$. Finally, making use of (1), we conclude that $x_m \in N_b$, $f_0(0) = 0$ and $f_N(1) = 1$. In consequence (18) holds, $(x_k)_{k\in\mathbb{N}} \in Z_b$ and $(y_k)_{k\in\mathbb{N}} \notin Z_b$. Moreover, if we assumed that x has a third address $(z_k)_{k\in\mathbb{N}}$, different from both of the first ones, we would have $(z_k)_{k\in\mathbb{N}} \in Z_b \setminus \{(x_k)_{k\in\mathbb{N}}\}$, which is impossible.

(ii) Assume that (18) holds and let a point $x \in A_*$ has an address $(x_k)_{k\in\mathbb{N}} \in Z_b$. Then there is $m \in N$ such that $x_m \in N_b$ and $x = f_{x_1,\ldots,x_m}(\mathbf{1})$. Applying now (18) we get

$$x = f_{x_1,\ldots,x_m}(1) = f_{x_1,\ldots,x_m+1}(0) = f_{x_1,\ldots,x_m+1}(\mathbf{0}),$$

which shows that (8) has an address not belonging to the set Z_b.

(iii) Assertion (i) implies that if (18) does not hold, then every point from A_* has exactly one address.

Assume now that every point from A_* has exactly one address and suppose that, on the contrary, (18) holds. Then $0 = \mathbf{0}$ and $1 = \mathbf{1}$. Fix $n \in N_b$ and put $x = f_n(\mathbf{1})$. Then $x \in A_*$ and $f_n(\mathbf{1}) = f_n(1) = f_{n+1}(0) = f_{n+1}(\mathbf{0})$, which jointly with (8) implies that x has two different addresses, a contradiction. $\square$

We now define a map $\pi\colon \{0,\dots,N\}^{\mathbb{N}}\setminus Z_b \to A_*$ by putting

$$\pi((x_k)_{k\in\mathbb{N}}) = \lim_{k\to\infty} f_{x_1,\dots,x_k}(0),$$

where Z_b is as in Lemma 5.3 in the case where (18) holds and $Z_b = \emptyset$ in the case where (18) does not hold.

Lemma 5.4. *The map π is a bijection.*

Proof. It is enough to apply Lemma 5.3 and [1, Theorem 1 in Chapter 4.2]. $\square$

Denote by σ the *Bernoulli shift*, i.e. the map from $\{0,\dots,N\}^{\mathbb{N}}$ into itself defined by

$$\sigma\big((x_k)_{k\in\mathbb{N}}\big) = (x_{k+1})_{k\in\mathbb{N}}.$$

Lemma 5.5. *For every $n \in \mathbb{N}$ we have*

$$\sigma^{-n}\circ\pi^{-1} = \pi^{-1}\circ T^{-n}.$$

Proof. We begin with proving that we have

$$\pi\circ\sigma = T\circ\pi \tag{19}$$

on $\{0,\dots,N\}^{\mathbb{N}}\setminus Z_b$.

First, we note that $\sigma(\{0,\dots,N\}^{\mathbb{N}}\setminus Z_b) \subset \{0,\dots,N\}^{\mathbb{N}}\setminus Z_b$. Fix $(x_k)_{k\in\mathbb{N}} \in \{0,\dots,N\}^{\mathbb{N}}\setminus Z_b$ and put $z = \lim_{k\to\infty} f_{x_2,\dots,x_k}(0)$. Then

$$\pi(\sigma((x_k)_{k\in\mathbb{N}})) = \lim_{k\to\infty} f_{x_2,\dots,x_k}(0) = z,$$

and

$$T(\pi((x_k)_{k\in\mathbb{N}})) = T\left(\lim_{k\to\infty} f_{x_1,\dots,x_k}(0)\right) = T\left(f_{x_1}(z)\right) = f^{-1}_{n(f_{x_1}(z))}(f_{x_1}(z)).$$

Since $z \in A_*$, we have $f_{x_1}(z) \in f_{x_1}(A_*)$, and so

$$x_1 \le n(f_{x_1}(z)).$$

Suppose for a contradiction that $x_1 < n(f_{x_1}(z))$. Then $f_{x_1}(z) \in f_{x_1+1}(A_*)$, and by (1) we have $z = 1$. Therefore, $f_{x_1}(1) = f_{x_1+1}(0)$ and $x_k = N$ for every $k \ge 2$, which is impossible as $(x_k)_{k\in\mathbb{N}} \notin Z_b$. In consequence $x_1 = n(f_{x_1}(z))$. Hence $f^{-1}_{n(f_{x_1}(z))}(f_{x_1}(z)) = z$, which yields that (19) holds.

To complete the proof it is enough to proceed by induction with the use of (19). $\square$

Let us consider now the measure $\mathbb{P}_{p_0,\dots,p_N}$ defined on $\{0,\dots,N\}$ by

$$\mathbb{P}_{p_0,\dots,p_N}(\{k\}) = p_k.$$

Note that $\mathbb{P}_{p_0,\dots,p_N}$ is a probability measure by (12). Further, we let $\mathbb{P}$ be the product measure on $\{0,\dots,N\}^{\mathbb{N}}$ of $\mathbb{N}$ copies of the measure $\mathbb{P}_{p_0,\dots,p_N}$. It is known that the Bernoulli shift is strong-mixing for $\mathbb{P}$ (see e.g. [2, Problem 4.3] or [3, Exercise 2.7.9]).

Lemma 5.6. *If $B \subset \{0,\dots,N\}^{\mathbb{N}} \setminus Z_b$ is a Borel set, then $\pi(B)$ is a Borel set and*

$$\mathbb{P}(B) = \mu(\pi(B)). \tag{20}$$

Proof. We first prove that if $B \subset \{0,\dots,N\}^{\mathbb{N}} \setminus Z_b$ is a Borel set, then $\pi(B)$ is a Borel set as well.

As every Borel set in $\{0,\dots,N\}^{\mathbb{N}} \setminus Z_b$ is generated by sets of the form

$$B = \left(\{x_1\} \times \cdots \times \{x_m\} \times \{0,\dots,N\}^{\mathbb{N}}\right) \setminus Z_b, \tag{21}$$

where $m \in \mathbb{N}$ and $x_1,\dots,x_m \in \{0,\dots,N\}$, it is sufficient to show that $\pi(B)$ is a Borel set for every set of the form (21).

Fix a set B of the form (21) with $m \in \mathbb{N}$ and $x_1,\dots,x_m \in \{0,\dots,N\}$. If either (18) does not hold or (18) holds and $x_m \notin N_b$, then by Lemma 5.4 we have

$$\pi(B) = f_{x_1,\dots,x_m}\left(\pi\left(\{0,\dots,N\}^{\mathbb{N}} \setminus Z_b\right)\right) = f_{x_1,\dots,x_m}(A_*). \tag{22}$$

If (18) holds and $x_m \in N_b$, then by Lemma 5.4 we have

$$\begin{aligned} \pi(B) &= f_{x_1,\dots,x_m}\left(\pi\left((\{0,\dots,N\}^{\mathbb{N}} \setminus Z_b)\setminus\{N\}^{\mathbb{N}}\right)\right) \\ &= f_{x_1,\dots,x_m}(A_*) \setminus f_{x_1,\dots,x_m}(\{\mathbf{1}\}). \end{aligned} \tag{23}$$

Since A_* and $\{\mathbf{1}\}$ are compact sets and $f_0,\dots,f_N$ are contractions, it follows that $f_{x_1,\dots,x_m}(A_*)$ and $f_{x_1,\dots,x_m}(\{\mathbf{1}\})$ are compact sets. In consequence, we see that in both of the considered cases the set $\pi(B)$ is Borel.

Now we prove that (20) holds for every Borel set $B \subset \{0,\dots,N\}^{\mathbb{N}} \setminus Z_b$.

Since every two Borel probability measures defined on $\{0,\dots,N\}^{\mathbb{N}} \setminus Z_b$ agreeing on cylinders are equal, it suffices to show that (20) holds for every cylinder $B \subset \{0,\dots,N\}^{\mathbb{N}} \setminus Z_b$. Moreover, by the additivity of the measures, we only need to show that (20) holds for every set of the form (21), where $m \in \mathbb{N}$ and $x_1,\dots,x_m \in \{0,\dots,N\}$.

Fix a set B of the form (21) with $m \in \mathbb{N}$ and $x_1,\dots,x_m \in \{0,\dots,N\}$. Then either (22) or (23) is satisfied, and by Lemmas 4.3 and 4.2 we see that in both cases we have

$$\mu\left(f^{-1}_{x_1,\dots,x_m}(\pi(B))\right) = \mu(A_*) = 1.$$

This jointly with (17) yields

$$\mu(\pi(B)) = \mu\left(f_{x_1}\left(\cdots\left(f_{x_m}\left(f^{-1}_{x_1,\dots,x_m}(\pi(B))\right)\right)\cdots\right)\right) = \prod_{i=1}^{m} p_{x_i}.$$

Finally, note that $\mathbb{P}(B) = \prod_{i=1}^{m} p_{x_i}$. □

Lemma 5.7. *The transformation T is strong-mixing for μ.*

Proof. The transformation T is measure preserving for μ by Lemma 5.2. To prove that it is strong-mixing for μ, fix two Borel sets $A, B \subset A_*$. Then using Lemmas 5.4 and 5.6, the fact that the Bernoulli shift is strong-mixing for $\mathbb{P}$ and Lemma 5.5 we get

$$\begin{aligned}\mu(A)\mu(B) &= \mu(\pi(\pi^{-1}(A)))\mu(\pi(\pi^{-1}(B))) = \mathbb{P}(\pi^{-1}(A))\mathbb{P}(\pi^{-1}(B))\\ &= \lim_{m\to\infty} \mathbb{P}(\sigma^{-m}(\pi^{-1}(A)) \cap \pi^{-1}(B)) = \lim_{m\to\infty} \mathbb{P}(\pi^{-1}(T^{-m}(A)) \cap \pi^{-1}(B))\\ &= \lim_{m\to\infty} \mathbb{P}(\pi^{-1}(T^{-m}(A) \cap B)) = \lim_{m\to\infty} \mu(T^{-m}(A) \cap B).\end{aligned}$$

The proof is complete. □

Denote by $\mathcal{M}^T(A_*)$ the set of all Borel probability measures defined on the σ-algebra of all Borel subsets of the interval $[0, 1]$ supported on A_*, making the transformation T measure preserving. Note that $\mu \in \mathcal{M}^T(A_*)$ by Lemma 4.2.

Lemma 5.8. *Every family of pairwise mutually singular measures belonging to the set $\mathcal{M}^T(A_*)$ is linearly independent.*

Proof. Fix $m \in \mathbb{N}\setminus\{1\}$, pairwise mutually singular measures $\mu_1, \ldots, \mu_m \in \mathcal{M}^T(A_*)$, numbers $\alpha_1, \ldots, \alpha_m \in \mathbb{R}\setminus\{0\}$ and assume by contradiction that

$$\sum_{i=1}^{m} \alpha_i \mu_i = 0.$$

Since the measures are mutually singular, for each $i, j \in \{1, \ldots, m\}$ with $i \neq j$ there are sets A_i^j and $A_j^i = X\setminus A_i^j$ such that $\mu_i(A_i^i) = \mu_j(A_j^i) = 0$. Put $A_m = \bigcup_{i=1}^{m-1} A_m^i$. Then

$$0 \leq \mu_m(A_m) = \mu_m\left(\bigcup_{i=1}^{m-1} A_m^i\right) \leq \sum_{i=1}^{m-1} \mu_m(A_m^i) = 0$$

and for every $j \in \{1, \ldots, m-1\}$ we have

$$1 \geq \mu_j(A_m) = \mu_j\left(X\setminus \bigcap_{i=1}^{m-1} A_i^m\right) \geq \mu_j(X\setminus A_j^m) \geq \mu_j(X) - \mu_j(A_j^m) = 1.$$

In consequence

$$\alpha_m = \sum_{i=1}^{m} \alpha_i \mu_i(X\setminus A_m) = 0,$$

and the proof is complete. □

Now we are in a position to prove that the set $\mathcal{W}$ is linearly independent.

Fix m different functions from $\mathcal{W}$ and consider the corresponding measures $\mu_1, \ldots, \mu_m \in \mathcal{M}^T(A_*)$. From Lemma 5.7 we infer that the transformation T is ergodic for all the measures. Thus $\mu_1, \ldots, \mu_m$ are extreme points of the set $\mathcal{M}^T(A_*)$ (see [3, Theorem 4.4] or [12, Proposition 12.4]), and hence they are pairwise mutually singular. Invoking Lemma 5.8 gives the claim.

5.2. An application of Theorem 5.1

In [9] Janusz Matkowski posed a problem asking about the existence of non-linear monotonic and continuous solutions $\Phi\colon [0,1] \to \mathbb{R}$ of a very particular case of the equation

$$\Phi(x) = \sum_{n=0}^{N} \Phi(f_n(x)) - \sum_{n=1}^{N} \Phi(f_n(0)). \tag{24}$$

Motivated by this problem, denote by $\mathcal{M}$ the vector space spanned by $\mathcal{W} \cup \{\mathbb{1}\}$, where $\mathbb{1}$ denotes the constant function that equals 1 on $[0,1]$. Note that by Theorem 5.1 and the fact that $\phi(0) = 0$ for each $\phi \in \mathcal{W}$, the set $\mathcal{W} \cup \{\mathbb{1}\}$ is a basis of $\mathcal{M}$.

Proposition 5.9. *Every function belonging to $\mathcal{M}$ is a continuous solution of Eq. (24). Moreover, if $\phi_1, \dots, \phi_m \in \mathcal{W}$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ are of the same sign, then the function $\sum_{i=1}^{m} \alpha_i \phi_i + \alpha_0$ is monotone for every $\alpha_0 \in \mathbb{R}$.*

Proof. Fix $\Phi \in \mathcal{M}$. Then there exist $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ and $\phi_1, \dots, \phi_m \in \mathcal{W}$ such that $\Phi = \sum_{i=1}^{m} \alpha_i \phi_i + \alpha_0$. Obviously, Φ is continuous. According to the first assertion of Lemma 2.2 we see that $\phi_i(f_0(0)) = 0$ for every $i \in \{1, \dots, m\}$, and hence applying also (E), we obtain

$$\begin{aligned}
\sum_{n=0}^{N} \Phi(f_n(x)) - \sum_{n=1}^{N} \Phi(f_n(0)) &= \sum_{n=0}^{N} \left(\sum_{i=1}^{m} \alpha_i \phi_i(f_n(x)) + \alpha_0 \right) \\
&\quad - \sum_{n=1}^{N} \left(\sum_{i=1}^{m} \alpha_i \phi_i(f_n(0)) + \alpha_0 \right) \\
&= \sum_{i=1}^{m} \alpha_i \left(\sum_{n=0}^{N} \phi_i(f_n(x)) - \sum_{n=0}^{N} \phi_i(f_n(0)) \right) + \alpha_0 \\
&= \sum_{i=1}^{m} \alpha_i \phi_i(x) + \alpha_0 = \Phi(x)
\end{aligned}$$

for every $x \in [0,1]$.

The moreover part of the assertion is clear. □

Acknowledgements

The research of both authors was supported by the University of Silesia Mathematics Department (Iterative Functional Equations and Real Analysis program). Furthermore, the research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement No. 291497 while the second author was a postdoctoral researcher at the University of Warwick.

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Received: January 23, 2018