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On orthogonally additive injections and surjections

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Summary. Let E be a real inner product space of dimension at least 2 and V a linear topological Hausdorff space. If $\operatorname{card} E \leqslant \operatorname{card} V$, then the set of all orthogonally additive injections mapping E into V is dense in the space of all orthogonally additive functions from E into V with the Tychonoff topology. If $\operatorname{card} V \leqslant \operatorname{card} E$, then the set of all orthogonally additive surjections mapping E into V is dense in the space of all orthogonally additive functions from E into V with the Tychonoff topology.

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Dedicated to Professor Henryk Hudzik
on his 70th birthday.

1. Introduction

Let E be a real inner product space of dimension at least 2 and V a linear topological Hausdorff space.

A function f mapping E into V is called orthogonally additive, if

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in E \text{ with } x \perp y.$$

It is well known (see [6, Corollary 10] and [4, Theorem 1]) that every orthogonally additive function f defined on E has the form

$$f(x) = a(\|x\|^2) + b(x) \quad \text{for } x \in E, \tag{1}$$

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where a and b are additive functions uniquely determined by f. Consequently, we have an operator Λ which to any orthogonally additive $f\colon E \to V$ assigns a pair (a, b) of additive functions such that (1) holds, i.e.

$$\Lambda f = (a, b) \tag{2}$$

where

$$a\colon \mathbb{R} \to V, \quad b\colon E \to V \text{ are additive and (1) holds.} \tag{3}$$

Putting

$$\mathrm{Hom}_{\perp}(E, V) = \{f\colon E \to V \,:\, f \text{ is orthogonally additive}\}$$

and

$$\mathrm{Hom}(S, V) = \{f\colon S \to V \,:\, f \text{ is additive}\}$$

for $S \in \{\mathbb{R}, E\}$, we see that $\Lambda\colon \mathrm{Hom}_{\perp}(E, V) \to \mathrm{Hom}(\mathbb{R}, V) \times \mathrm{Hom}(E, V)$, given by (2) and (3), is a linear bijection.

Given a non-empty set S, consider the set V^S of all functions from S into V with the usual Tychonoff topology; clearly V^S is a linear topological space. In what follows, we consider $\mathrm{Hom}_{\perp}(E, V)$ and $\mathrm{Hom}(S, V)$ for $S \in \{\mathbb{R}, E\}$ with the topology induced by V^E and V^S, respectively. According to [3, Theorem 1]:

1. *The isomorphism* $\Lambda\colon \mathrm{Hom}_{\perp}(E, V) \to \mathrm{Hom}(\mathbb{R}, V) \times \mathrm{Hom}(E, V)$ *given by* (2) *and* (3) *is a homeomorphism.*

As an immediate consequence of [2, Theorem] and [3, Corollary 1] we have the following information:

2. *The set*

$$\{f \in \mathrm{Hom}_{\perp}(E, V)\colon f \textit{ is injective and } f(E) = V\}$$

is nowhere dense in $\mathrm{Hom}_{\perp}(E, V)$.

Basing on the continuity of Λ^{-1} and ideas from [1, Remarks 1 and 2], we are going to show that each of the sets

$$\{f \in \mathrm{Hom}_{\perp}(E, V) \,:\, f \text{ is injective}\}, \tag{4}$$

$$\{f \in \mathrm{Hom}_{\perp}(E, V) \,:\, f(E) = V\} \tag{5}$$

is dense in $\mathrm{Hom}_{\perp}(E, V)$ (cardinalities of E and V permitting).

2. Results

They read as follows.

2.1. Theorem. *If* $\operatorname{card} E \leqslant \operatorname{card} V$, *then set* (4) *is dense in* $\mathrm{Hom}_{\perp}(E, V)$.

2.2. Theorem. *If* $\operatorname{card} V \leqslant \operatorname{card} E$, *then set* (5) *is dense in* $\mathrm{Hom}_{\perp}(E, V)$.

Since the intersection of a dense set with the complement of a nowhere dense set is dense, the above theorems imply the following corollaries.

2.3. Corollary. *If* $\operatorname{card} E \leqslant \operatorname{card} V$*, then the set*

$$\{f \in \operatorname{Hom}_{\perp}(E, V) \,:\, f \text{ is injective and } f(E) \neq V\}$$

is dense in $\operatorname{Hom}_{\perp}(E, V)$.

2.4. Corollary. *If* $\operatorname{card} V \leqslant \operatorname{card} E$*, then the set*

$$\{f \in \operatorname{Hom}_{\perp}(E, V) \,:\, f(E) = V \text{ and } f \text{ is not injective}\}$$

is dense in $\operatorname{Hom}_{\perp}(E, V)$.

3. Proofs

We start with two lemmas.

3.1. Lemma. *If* $\operatorname{card} E \leqslant \operatorname{card} V$*, then the set*

$$\{(a, b) \in \operatorname{Hom}(\mathbb{R}, V) \times \operatorname{Hom}(E, V) \,:\, a(\mathbb{R}) \cap b(E) = \{0\} = b^{-1}(\{0\})\} \tag{6}$$

is dense in $\operatorname{Hom}(\mathbb{R}, V) \times \operatorname{Hom}(E, V)$.

To formulate the second lemma, denote by $\mathscr{L}$ the set of all one-dimensional linear subspaces of E and by $\mathscr{V}$ the set of all pairs (V_1, V_2) of linear subspaces of V over the field $\mathbb{Q}$ of all rationals such that V_1 is finite-dimensional and $V_1 \oplus V_2 = V$.

Observe that if $L \in \mathscr{L}$, then $E = L \oplus L^{\perp}$, whence $\operatorname{card} E = \mathfrak{c} \cdot \operatorname{card} L^{\perp}$, and this implies (see, e.g., [7, formula (2.1) on p. 414]) that

$$\operatorname{card} E = \operatorname{card} L^{\perp}. \tag{7}$$

3.2. Lemma. *If* $\operatorname{card} V \leqslant \operatorname{card} E$*, then the set*

$$\bigcup_{L \in \mathscr{L}, (V_1, V_2) \in \mathscr{V}} \{(a, b) \in \operatorname{Hom}(\mathbb{R}, V) \times \operatorname{Hom}(E, V) \,:\, a([0, \infty)) = V_1,\ V_2 \subset b(L^{\perp}),\ b(L) = \{0\}\} \tag{8}$$

is dense in $\operatorname{Hom}(\mathbb{R}, V) \times \operatorname{Hom}(E, V)$.

Proof of the Lemmas. We divide it into three parts.

Part I. It concerns both lemmas. Fix non-empty open sets $\mathcal{U} \subset \mathrm{Hom}(\mathbb{R}, V)$ and $\mathcal{V} \subset \mathrm{Hom}(E, V)$. To show that $\mathcal{U} \times \mathcal{V}$ intersects the considered set (i.e. (6) or (8)), we may assume that $V \neq \{0\}$,

$$\mathcal{U} = a_0 + \left\{a \in \mathrm{Hom}(\mathbb{R}, V) : a(\alpha_n) \in U \text{ for } n \in \{1, \ldots, N\}\right\}$$

and

$$\mathcal{V} = b_0 + \left\{b \in \mathrm{Hom}(E, V) : b(x_n) \in U \text{ for } n \in \{1, \ldots, N\}\right\}$$

with $a_0 \in \mathrm{Hom}(\mathbb{R}, V)$, $b_0 \in \mathrm{Hom}(E, V)$, a neighbourhood U of zero in V and some $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$, $x_1, \ldots, x_N \in E$, $N \in \mathbb{N}$.

Let $H_{\mathbb{R}} \subset (0, \infty)$ and H_E be bases of the vector spaces $\mathbb{R}$ and E, respectively, both of them over $\mathbb{Q}$. There are finite sets $H_{\mathbb{R}}^0 \subset H_{\mathbb{R}}$ and $H_E^0 \subset H_E$ such that $\alpha_n \in \mathrm{Lin}_{\mathbb{Q}} H_{\mathbb{R}}^0$ and $x_n \in \mathrm{Lin}_{\mathbb{Q}} H_E^0$ for $n \in \{1, \ldots, N\}$. For every $n \in \{1, \ldots, N\}$, $\alpha \in H_{\mathbb{R}}^0$ and $x \in H_E^0$ let $\rho_\alpha^{(n)}$ and $r_x^{(n)}$ be rationals such that

$$\alpha_n = \sum_{\alpha \in H_{\mathbb{R}}^0} \rho_\alpha^{(n)} \alpha, \quad x_n = \sum_{x \in H_E^0} r_x^{(n)} x. \tag{9}$$

Choose now a neighbourhood U_0 of zero in V such that

$$\sum_{\alpha \in H_{\mathbb{R}}^0} \rho_\alpha^{(n)} U_0 \cup \sum_{x \in H_E^0} r_x^{(n)} U_0 \subset U \quad \text{for } n \in \{1, \ldots, N\}, \tag{10}$$

and injective functions $a_1 \colon H_{\mathbb{R}}^0 \to V,\ b_1 \colon H_E^0 \to V$ such that

$$a_1(H_{\mathbb{R}}^0) \cup b_1(H_E^0) \text{ is linearly independent over } \mathbb{Q}, \quad a_1(H_{\mathbb{R}}^0) \cap b_1(H_E^0) = \varnothing$$

and

$$a_1(\alpha) \in U_0 + a_0(\alpha) \text{ for } \alpha \in H_{\mathbb{R}}^0, \quad b_1(x) \in U_0 + b_0(x) \text{ for } x \in H_E^0. \tag{11}$$

Part II. It concerns Lemma 3.1. Let H be a basis of V over $\mathbb{Q}$ containing $a_1(H_{\mathbb{R}}^0) \cup b_1(H_E^0)$. Since (see, e.g., the proof of [5, Lemma 4.2.2])

$$\begin{aligned}\mathrm{card}(H_E \backslash H_E^0) &= \mathrm{card}\, H_E = \mathrm{card} \operatorname*{Lin}_{\mathbb{Q}} H_E = \mathrm{card}\, E \\ &\leqslant \mathrm{card}\, V = \mathrm{card} \operatorname*{Lin}_{\mathbb{Q}} H = \mathrm{card}\, H = \mathrm{card}\big(H \backslash (a_1(H_{\mathbb{R}}^0) \cup b_1(H_E^0))\big),\end{aligned}$$

we may extend b_1 to an injective $b_2 \colon H_E \to H \backslash a_1(H_{\mathbb{R}}^0)$. Let $a_2 \colon H_{\mathbb{R}} \to a_1(H_{\mathbb{R}}^0)$ be an extension of a_1 and consider the additive functions $a \colon \mathbb{R} \to V$ and $b \colon E \to V$ such that $a|_{H_{\mathbb{R}}} = a_2$ and $b|_{H_E} = b_2$. Clearly, b is injective,

$$\begin{aligned}a(\mathbb{R}) \cap b(E) &= \operatorname*{Lin}_{\mathbb{Q}} a(H_{\mathbb{R}}) \cap \operatorname*{Lin}_{\mathbb{Q}} b(H_E) \\ &= \operatorname*{Lin}_{\mathbb{Q}} a_2(H_{\mathbb{R}}) \cap \operatorname*{Lin}_{\mathbb{Q}} b_2(H_E) \subset \operatorname*{Lin}_{\mathbb{Q}} a_1(H_{\mathbb{R}}^0) \cap \operatorname*{Lin}_{\mathbb{Q}}(H \backslash a_1(H_{\mathbb{R}}^0)) = \{0\},\end{aligned}$$

and, according to (9), (11) and (10), for any $n \in \{1, \dots, N\}$ we have

$$a(\alpha_n) - a_0(\alpha_n) = \sum_{\alpha \in H^0_{\mathbb{R}}} \rho^{(n)}_\alpha (a_1(\alpha) - a_0(\alpha)) \in \sum_{\alpha \in H^0_{\mathbb{R}}} \rho^{(n)}_\alpha U_0 \subset U \tag{12}$$

and

$$b(x_n) - b_0(x_n) = \sum_{x \in H^0_E} r^{(n)}_x (b_1(x) - b_0(x)) \in \sum_{x \in H^0_E} r^{(n)}_x U_0 \subset U. \tag{13}$$

It shows that (a, b) is in the set (6) and in $\mathcal{U} \times \mathcal{V}$.

Part III. It concerns Lemma 3.2. Let L be a one-dimensional linear subspace of E such that

$$L \cap \operatorname*{Lin}_{\mathbb{Q}} H^0_E = \{0\}.$$

Let H_L be a basis of L over $\mathbb{Q}$, and $H_{L^\perp}$ a basis of $L^\perp$ over $\mathbb{Q}$. Put

$$V_1 = \operatorname*{Lin}_{\mathbb{Q}} a_1(H^0_{\mathbb{R}})$$

and let $a: \mathbb{R} \to V_1$ be an additive extension of a_1 such that $a([0, \infty)) = V_1$. Consider now a basis H of V over $\mathbb{Q}$ containing $a_1(H^0_{\mathbb{R}})$. Then, by (7),

$$\operatorname{card}(H \backslash a_1(H^0_{\mathbb{R}})) = \operatorname{card} H = \operatorname{card} V \leqslant \operatorname{card} E = \operatorname{card} L^\perp = \operatorname{card} H_{L^\perp} = \operatorname{card}(H_{L^\perp} \backslash H^0_E),$$

and so there is an additive extension $b: E \to V$ of b_1 such that

$$b(H_L) = \{0\} \quad \text{and} \quad b(H_{L^\perp} \backslash H^0_E) = H \backslash a_1(H^0_{\mathbb{R}}).$$

Putting

$$V_2 = \operatorname*{Lin}_{\mathbb{Q}}(H \backslash a_1(H^0_{\mathbb{R}})),$$

we see that $(V_1, V_2) \in \mathscr{V}$. It shows that (a, b) is in the set (8). Moreover, by (9), (11) and (10), for every $n \in \{1, \dots, N\}$ we have (12) and (13) and so (a, b) is also in $\mathcal{U} \times \mathcal{V}$. □

Proof of Theorem 2.1. Denote the set (6) by $\mathcal{D}$. Since Λ^{-1} is continuous, it follows from Lemma 3.1 that $\Lambda^{-1}(\mathcal{D})$ is dense in $\operatorname{Hom}_\perp(E, V)$, and it is enough to show that any f in $\Lambda^{-1}(\mathcal{D})$ is injective. Indeed, let $f \in \Lambda^{-1}(\mathcal{D})$. Then

$$(a, b) := \Lambda f \in \mathcal{D}, \tag{14}$$

and if $f(x) = f(y)$ for some $x, y \in E$, then

$$a(\|x\|^2) - a(\|y\|^2) = b(y) - b(x).$$

The left-hand side belongs to $a(\mathbb{R})$ and the right-hand side is in $b(E)$, whence $b(x) = b(y)$ and $x = y$. □

Proof of Theorem 2.2. Denoting now by $\mathcal{D}$ the set (8), it is enough to show that any f in $\Lambda^{-1}(\mathcal{D})$ maps E onto V. Indeed, let $f \in \Lambda^{-1}(\mathcal{D})$. Keeping (14), we can find $L \in \mathscr{L}$ and $(V_1, V_2) \in \mathscr{V}$ such that

$$a([0,\infty)) = V_1,\ V_2 \subset b(L^\perp) \quad \text{and} \quad b(L) = \{0\}.$$

Fix arbitrary $v \in V$. Then $v = v_1 + v_2$ with $v_1 \in V_1$, $v_2 \in V_2$, and $v_2 = b(x_2)$ for some $x_2 \in L^\perp$. Choose $\alpha \in [0,\infty)$ with

$$a(\alpha) = v_1 - a(\|x_2\|^2)$$

and then $x_1 \in L$ with $\|x_1\|^2 = \alpha$. We have

$$f(x_1 + x_2) = a(\|x_1\|^2 + \|x_2\|^2) + b(x_1 + x_2) = a(\alpha) + a(\|x_2\|^2) + b(x_2) = v_1 + v_2 = v,$$

which ends the proof. □

The reader interested in further problems connected with orthogonal additivity is referred to a survey paper [8] by Justyna Sikorska.

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