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On the assignability of regularity coefficients and central exponents of discrete linear time-varying systems

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Abstract: In this paper we investigate the problem of assignability of the so-called regularity coefficients and central exponents of discrete linear time-varying systems. The main result presents a possibility of assignability of Lyapunov, Perron, Grobman regularity coefficients and central exponents by a linear time-varying feedback under the assumptions of uniform complete controllability.

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1. INTRODUCTION

When we consider controlled dynamic system then an important question is to describe conditions that guarantee that certain properties of the system may be achieved by a feedback control. One may, for example, stabilize the system or more generally obtain a given rate of convergency of solutions, reduce oscillation, minimize certain cost functional, size of overshoot etc.. Such desired properties are sometimes described by certain numerical characteristics such as Lyapunov, Bohl, Perron, central and general exponents (see, Czornik (2012)) and then we may formulate the problem of achieving the desired dynamic properties as a problem of location of the appropriate numerical characteristic in a given position. A typical example of this methodology is pole placement theorem for linear time-invariant systems which says that the controllability is equivalent to the fact that for each set of complex numbers with cardinality equal to the dimension of the state vector and symmetric relative to the real axis, there is a stationary feedback such that the poles of the closed-loop system form this set (see Dickinson (1974)). At the same time, for such systems, relationships between the location of the poles and dynamic properties of the system such as stability, stability and oscillation degrees, and the size of overshoot are well known and described in the literature Antsaklis and Michel (2006); Boyd and Barratt (1991). Generalization of this methodology to the time-varying systems is presented in Popova (2004) for continuous-time systems and Babiarz et al. (2017, 2019, 2018) for discrete-time systems.

In this paper we investigate the problem of assignability of the so-called regularity coefficients and central exponents for discrete linear time-varying systems. The notion of regularity was firstly introduced for continuous linear time-varying systems in Lyapunov (1892) (see also Lyapunov (1956)) and this concept was further developed in Vinograd (1956). For discrete time systems this property was investigated in Barreira and Valls (2007, 2016); Czornik and Nawrat (2010). The importance of the regularity notion in the linear approximation stability theory relies on the following fundamental fact: if the system of linear approximation of a certain nonlinear system is regular and has negative Lyapunov exponents then the nonlinear system is exponentially stable (Lyapunov (1892)). The notion of regularity also plays a major role in smooth ergodic theory, in which case certain integrability assumptions guarantee that the linearizations along almost all trajectories are Lyapunov regular (Barreira and Pesin (2002)). Moreover, regular systems have the following important properties: the Lyapunov exponents of regular systems are sharp (i.e. the upper limit in the definition is in fact the limit); the Lyapunov spectrum of the adjoint system is symmetric with respect to zero to the spectrum of the initial system; for an upper triangular system its regularity is equivalent to the existence of finite average values of the diagonal elements; the transition matrix of a regular systems has a special form, which enables to extend the notion of regularity to systems with unbounded coefficients;

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the Lyapunov transformation preserves the regularity of a system (Adrianova (1995); Czornik (2012)).

The property of regularity may be defined by the so-called regularity coefficients. In the literature three basic definitions of regularity coefficients can be found: Lyapunov, Perron and Grobman. Below we describe relations between them. It can be shown that if one of them is equal to zero, then so are the others. Regular systems are those for which the regularity coefficient is equal to zero. Regularity coefficients, in addition to defining regular systems, are used to describe their dynamic properties. In Czornik and Nawrat (2011) it was shown how the Grobman regularity coefficient describes the effect of parametric inaccuracies on the values of the Lyapunov exponents. In turn, Barreira and Valls (2007) show how the regularity coefficients can be used to describe the effect of parametric disturbances on the property of non-uniform dichotomy. In Czornik et al. (2019) a complete description of all possible relations between regularity coefficients of the system and its adjoint were presented.

When we build a mathematical model of a real system we usually assume that we know the exact values of system parameters. However, in practice this assumption is not always satisfied for a variety of reasons, for example, difficulty in obtaining accurate model estimates or time variation of plant parameters. Therefore the determination of the movability boundaries of the Lyapunov spectrum under various perturbations is one of the main problems of the theory of the Lyapunov exponents (Bylov et al. (1966)). This problem has been studied for continuous-time systems for many classes of parametric perturbations. The monograph (Izobov (2013)) is almost completely devoted to this problem for continuous-time systems and monograph by Czornik (2012) for discrete-time systems. To describe the changes of the greatest and smallest Lyapunov exponents under arbitrary small perturbation we use the central exponents (Czornik and Niezabitowski (2013)). In this paper we also investigate the problem of assignability of these quantities by a linear time-varying feedback.

2. NOTATION AND PROBLEM FORMULATION

We consider the discrete linear time-varying system

$$x(n+1) = A(n)x(n) + B(n)u(n), \tag{1}$$

where $A = (A(n))_{n\in\mathbb{N}}$, $B = (B(n))_{n\in\mathbb{N}}$ are sequences of s by s and s by t real matrices, respectively. We assume that A and B are bounded, A consists of invertible matrices and the sequence $A^{-1} = \left(A^{-1}(n)\right)_{n\in\mathbb{N}}$ is bounded. The control sequence $u = (u(n))_{n\in\mathbb{N}}$ is $t-$dimensional.

The solution of (1) corresponding to the control u and initial condition $x(k_0) = x_0$, where $k_0 \in \mathbb{N}$ and $x_0 \in \mathbb{R}^s$ is denoted by

$$\left(x\left(n, k_0, x_0, u\right)\right)_{n\geq k_0}$$

and is given by the following formula

$$x\left(n, k_0, x_0, u\right) =$$

$$\Phi_A\left(n, k_0\right) x_0 + \sum_{j=k_0}^{n-1} \Phi_A\left(n, j+1\right) B(j)u(j), \tag{2}$$

where $\Phi_A\left(n, k_0\right)$ is the transition matrix of

$$x(n+1) = A(n)x(n) \tag{3}$$

given by

$$\Phi_A(n,n) = I$$
$$\Phi_A(n,j) = A(n-1)\ldots A(j) \quad \text{for} \quad n > j.$$

Additionally we define

$$\Phi_A(j,n) = \Phi_A^{-1}(n,j) \quad \text{for} \quad n > j.$$

Together with the system (3) we will consider the adjoint system

$$y(n+1) = A^{-T}(n)y(n), \tag{4}$$

where $A^{-T}(\cdot) = (A^T(\cdot))^{-1}$. It is obvious that the system adjoint to system (4) is system (3), therefore systems (3) and (4) are called mutually adjoint.

By $\|\cdot\|$ we denote the Euclidean norm in $\mathbb{R}^s$ and the induced operator norm.

A bounded sequence $(D(n))_{n\in\mathbb{N}}$ of invertible s-by-s matrices such that $(D^{-1}(n))_{n\in\mathbb{N}}$ is bounded, will be called Lyapunov sequence. If $(D(n))_{n\in\mathbb{N}}$ is a Lyapunov sequence, then the transformation which transforms a sequence $(x(n))_{n\in\mathbb{N}}$ of elements of $\mathbb{R}^s$ into a sequence $(y(n))_{n\in\mathbb{N}}$ according to formula

$$y(n) = D(n)x(n) \tag{5}$$

will be called Lyapunov transformation.

The Lyapunov exponent $\lambda(a)$ of a sequence $a = (a(n))_{n\in\mathbb{N}}$ of real numbers is the number or symbol $\pm\infty$ defined by the formula

$$\lambda(a) = \limsup_{n\to\infty} n^{-1} \ln |a(n)|.$$

If $x = (x(n))_{n\in\mathbb{N}}$ is a sequence of vectors, then the Lyapunov exponent $\lambda(x)$ of this sequence is defined as the Lyapunov exponent of the sequence $(\|x(n)\|)_{n\in\mathbb{N}}$ consisting of the norms of the elements of the original sequence, i.e.

$$\lambda(x) = \limsup_{n\to\infty} n^{-1} \ln \|x(n)\|.$$

For an initial condition $x(0) = x_0 \in \mathbb{R}^s$ the solution of (3) is denoted by $x(n, x_0)$, so

$$x(n, x_0) = \Phi_A(n, 0)\, x_0, \ n \in \mathbb{N}. \tag{6}$$

For $x_0 \in \mathbb{R}^s$, $x_0 \neq 0$ the Lyapunov exponent $\lambda(x_0)$ of (6) is defined as

$$\lambda(x_0) = \limsup_{n\to\infty} \frac{1}{n} \ln \|x(n, x_0)\|\,.$$

It is well known (Barreira and Pesin (2002)) that the set of Lyapunov exponents of all nontrivial solutions of system (3) contains at most s elements. Moreover, if $A = (A\left(n\right))_{n\in\mathbb{N}}$ is a Lyapunov sequence, then all the Lyapunov exponents are finite and we denote them as follows

$$-\infty < \lambda_1'\left(A\right) < \lambda_2'\left(A\right) < \ldots < \lambda_r'\left(A\right) < \infty, \, r \leq s.$$

For each $\lambda_i'(A)$ we consider the following subspaces of $\mathbb{R}^s$

$$E_i = \{v \in \mathbb{R}^s : \lambda(v) \leq \lambda_i'(A)\}$$

and we set $E_0 = \{0\}$. The multiplicity n_i of the Lyapunov exponent $\lambda_i'(A)$ is defined as

$$\dim E_i - \dim E_{i-1}.$$

The sequence

$$\left(\lambda_1(A), \lambda_2(A), \ldots, \lambda_s(A)\right),$$

where each Lyapunov exponent $\lambda_i'(A)$ appears $n_i, i = 1, 2, ..., s$ times and $\lambda_i(A) \leq \lambda_{i+1}(A)$, $i = 1, 2, ..., s-1$ will be called the Lyapunov spectrum of (3).

Definition 1. A bounded sequence $U = (U(n))_{n\in\mathbb{N}}$ of $t-$by$-s$ matrices is said to be an admissible feedback control if $(A(n) + B(n)U(n))_{n\in\mathbb{N}}$ is a Lyapunov sequence.

For an admissible control the spectrum of the closed system

$$x(n+1) = (A(n) + B(n)U(n))\, x(n) \tag{7}$$

will be denoted by

$$(\lambda_1(A + BU), \lambda_2(A + BU), \ldots, \lambda_s(A + BU)).$$

With each system (3) we associate the so-called: Lyapunov regularity coefficient $\sigma_L(A)$, Perron regularity coefficient $\sigma_P(A)$ and Grobman regularity coefficient $\sigma_G(A)$(Lyapunov (1892); Perron (1930); Grobman (1952); Czornik (2012)). Now, we will present definitions of them. Let

$$\left(\lambda_1(A^{-T}), \lambda_2(A^{-T}), \ldots, \lambda_s(A^{-T})\right)$$

denote the Lyapunov spectrum of the adjoint system (4). By $\Psi(A)$ we denote the set of all fundamental matrices of the system (3). For any sequence $(X(n))_{n=0}^{+\infty}$ of $s-$by$-s$ matrices by $\lambda_i[X]$ we denote the Lyapunov exponent of its i-th column, $i = 1, \ldots, s$. The Lyapunov, Perron and Grobman regularity coefficients are given by the following formulae:

$$\sigma_L(A) = \sum_{i=1}^{s} \lambda_i(A) - \liminf_{n\to\infty} n^{-1} \ln|\det \Phi_A(n,0)|, \tag{8}$$

$$\sigma_P(A) = \max_{1\leq i\leq s} \left\{\lambda_i(A) + \lambda_{s-i+1}(A^{-T})\right\}, \tag{9}$$

$$\sigma_G(A) = \inf_{\Phi\in\Psi(A)} \max_{1\leq i\leq s} \left\{\lambda_i[\Phi] + \lambda_i\left[\Phi^{-T}\right]\right\}. \tag{10}$$

In Czornik et al. (2019) the following theorem, which describe relation between regularity coefficients, has been shown

Theorem 2. For any natural $s \geq 2$ *and an ordered quadruple* (p, g, l, l^*) *of real numbers, there exists a system (*3) such that

$$\sigma_P(A) = p,\ \ \sigma_G(A) = g,\ \ \sigma_L(A) = l$$

and

$$\sigma_L(A^{-T}) = l^*,$$

if and only if

$$0 \leq p \leq g \leq \min(l, l^*) \leq \max(l, l^*) \leq sp.$$

One of the main purpose of the paper is to investigate the conditions for the assignability of the regularity coefficients of system (7).

Together with (3) we consider the following perturbed system

$$z(n+1) = (A(n) + \Delta(n))\, z(n), \tag{11}$$

where $\Delta = (\Delta(n))_{n\in\mathbb{N}}$ is a sequence of s-by-s real matrices from a certain class $\mathfrak{M}$. Under the influence of the perturbation Δ, the Lyapunov spectrum of (3) vary, in general, discontinuously (see Czornik et al. (2010, 2012)). It is possible that a finite shift of the characteristic exponents of the original system (3) corresponds to an arbitrarily small $\sup\|\Delta(n)\|$. In particular, it is possible for an exponentially stable system to be perturbed by an exponentially decreasing perturbation and the resulting system is not stable. The quantities

$$\Lambda_g(\mathfrak{M}) = \sup\{\lambda_s(A+\Delta) : \Delta \in \mathfrak{M}\}, \tag{12}$$

$$\Lambda_s(\mathfrak{M}) = \inf\{\lambda_g(A+\Delta) : \Delta \in \mathfrak{M}\} \tag{13}$$

are referred to as the maximal upper and minimal lower movability boundary of the greatest exponent of (3) with perturbation in the class $\mathfrak{M}$. Analogically,

$$\lambda_g(\mathfrak{M}) = \sup\{\lambda_1(A+\Delta) : \Delta \in \mathfrak{M}\},$$

$$\lambda_s(\mathfrak{M}) = \inf\{\lambda_1(A+\Delta) : \Delta \in \mathfrak{M}\}$$

are referred to as the maximal upper and minimal lower movability boundary of the smallest exponent of (3). An important role in finding the movability boundaries for arbitrary small perturbation is played by upper $\Omega(A)$ and lower $\omega(A)$ central exponents (see Czornik and Niezabitowski (2013)) given by the following formulae

$$\Omega(A) = \lim_{N\to\infty} \limsup_{n\to\infty} \frac{1}{nN} \sum_{i=1}^{n} \ln\|\Phi(iN,(i-1)N)\| \tag{14}$$

and

$$\omega(A) = \lim_{N\to\infty} \liminf_{n\to\infty} \frac{1}{nN} \sum_{i=1}^{n} \ln\left\|\Phi^{-1}(iN,(i-1)N)\right\|^{-1}. \tag{15}$$

The role of these quantities in finding the movability boundaries is explained in the following theorems taken from Czornik and Niezabitowski (2013) and Czornik and Nawrat (2011).

Theorem 3. The following inequalities hold

$$\lim_{\delta\to 0^+}\left(\sup_{\|\Delta\|_\infty<\delta} \lambda_g(A+\Delta)\right) \leq \Omega(A)$$

and

$$\lim_{\delta\to 0^+}\left(\inf_{\|\Delta\|_\infty<\delta} \lambda_s(A+\Delta)\right) \geq \omega(A).$$

Theorem 4. If the perturbation $\Delta = (\Delta(n))_{n\in\mathbb{N}}$ in (11) is such that

$$\limsup_{n\to\infty} \frac{1}{n}\ln\|\Delta(n)\| < -\sigma_G(A),$$

then the spectra of (3) and (11) coincide.

In the rest of the paper we will investigate the conditions for the assignability of the central exponents and regularity coefficients of system (7).

We will show that conditions for assignability of regularity coefficients and central exponents are related to the controllability of the system (1). In the literature, a number of non-equivalent definitions of controllability for system (1) is considered. We have global controllability, controllability from zero, controllability to zero, and others (see Klamka (1991)). In the paper, we use the concept of uniform complete controllability (see (Halanay and Ionescu, 1994, p. 33), Zaitsev et al. (2014)).

Definition 5. The system (1) is uniformly completely controllable if there exist a positive γ and a natural number K such that for all $x_1 \in \mathbb{R}^s$ and $k_0 \in \mathbb{N}$, there exists a control sequence $\widehat{u}(n)$, $n = k_0, k_0+1, \ldots, k_0+K-1$ such that

$$x(k_0+K, k_0, 0, \widehat{u}) = x_1$$

and

$$\|\widehat{u}(n)\| \leq \gamma\|x_1\|$$

for all $n = k_0, k_0+1, \dots, k_0+K-1$. If K is given, we say that the system is K-uniformly completely controllable.

In the investigation of controllability, a crucial role is played by the following Kalman controllability matrix:

$$W(k,n) = \sum_{j=k}^{n-1} \Phi_A(k,j+1)B(j)B^T(j)\Phi_A^T(k,j+1),$$

for all $n > k$. Theorem 6 (see Halanay and Ionescu (1994), Proposition 3, page 34) gives a necessary and sufficient condition for K-uniform complete controllability in terms of the Kalman controllability matrix.

Theorem 6. Suppose that A is a Lyapunov sequence and B is bounded. Then the system (1) is K-uniformly completely controllable if and only if there exists $\gamma > 0$ such that

$$W(k_0, k_0+K) \geq \gamma I,$$

for all $k_0 \in \mathbb{N}$.

Let us recall definition of dynamically equivalent systems (see Halanay and Ionescu (1994)).

Definition 7. We say that the system (3) is dynamically equivalent to system

$$y(n+1) = C(n)y(n), \tag{16}$$

if there exists a Lyapunov sequence $(D(n))_{n\in\mathbb{N}}$ such that

$$C(n) = D^{-1}(n+1)A(n)D(n).$$

The notion of "dynamically equivalent systems" is well-known in the theory of linear differential and difference equations and has been used to reduce such equations to a simpler one (Adrianova (1995); Barreira and Pesin (2002)). In this approach a crucial role is played by the fact that equivalent systems share many dynamic properties, in particular dynamically equivalent systems have the same Lyapunov spectrum (Izobov (2013); Czornik (2012)) and therefore they have the same regularity coefficients as well the same central exponents. The paper Gohberg et al. (1952) systematically studies the dynamic equivalence problem for discrete linear time-varying systems without input. In our further considerations we will use the assumption about uniform complete controllability to find an admissible feedback control such that the closed-loop system is dynamically equivalent to a simpler system, namely diagonal of a particular form. If it is possible such a system is called scalarizable. The formal definition is as follows.

Definition 8. We say that the system (7) is globally positively scalarizable if for any Lyapunov sequence $p = (p(n))_{n\in\mathbb{N}}$ of positive real numbers there exists an admissible feedback control $U = (U(n))_{n\in\mathbb{N}}$ such that (7) is dynamically equivalent to the system (16) with $C(n) = p(n)I_s$, where I is the identity matrix of size s.

Globally scalarizable systems have been introduced in Popova (2004) as a tool to control the exponents. The next theorem states that uniform complete controllability is a sufficient condition for scalarizability. The proof of this theorem has been published in Babiarz et al. (2017).

Theorem 9. Suppose that A is a Lyapunov sequence, B is bounded and the system (1) is uniformly completely controllable. Then the system (7) is globally positively scalarizable.

3. MAIN RESULTS

The next theorem contains the first main result of this paper.

Theorem 10. If system (1) is uniformly completely controllable then for each $\sigma \geq 0$ there exists admissible feedback control $U = (U(n))_{n\in\mathbb{N}}$ such that for the closed loop system (7) we have

$$s\sigma = \sigma_L(A+BU) \tag{17}$$

and

$$\sigma = \sigma_P(A+BU) = \sigma_G(A+BU). \tag{18}$$

To prove it we will use the following lemma.

Lemma 11. We have

$$\limsup_{n\to\infty} \sin\ln(n) = 1. \tag{19}$$

$$\liminf_{n\to\infty} \sin\ln(n) = -1. \tag{20}$$

Proof. Consider the following sequence

$$t_k = \exp\left(2k+\frac{1}{2}\right)\pi,\ n_k = \lfloor t_k \rfloor,\ k\in\mathbb{N},$$

where

$$\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\}.$$

We have

$$1 \geq \limsup_{n\to\infty}\sin\ln(n) \geq \limsup_{k\to\infty}\sin\ln(n_k) =$$
$$\limsup_{k\to\infty}\sin\left(\ln(t_k)+\ln\left(\frac{n_k}{t_k}\right)\right) =$$
$$\limsup_{k\to\infty}\left(\sin(\ln(t_k))\cos\left(\ln\left(\frac{n_k}{t_k}\right)\right) + \right.$$
$$\left.\cos(\ln(t_k))\sin\left(\ln\left(\frac{n_k}{t_k}\right)\right)\right) =$$
$$\limsup_{k\to\infty}\left(1\cdot\cos\left(\ln\left(\frac{n_k}{t_k}\right)\right) + \right.$$
$$\left.0\cdot\sin\left(\ln\left(\frac{n_k}{t_k}\right)\right)\right) =$$
$$\limsup_{k\to\infty}\cos\left(\ln\left(\frac{n_k}{t_k}\right)\right) =$$
$$\cos\left(\ln\lim_{k\to\infty}\left(\frac{n_k}{t_k}\right)\right) = \cos\ln 1 = 1.$$

Considering sequences

$$t_k = \exp\left(2k+\frac{3}{2}\right)\pi,\ n_k = \lfloor t_k \rfloor,\ k\in\mathbb{N},$$

we may prove in a similar way the equality (20).

Proof. [Proof of Theorem 10] Consider system

$$y(n+1) = p(n)I_s y(n), \tag{21}$$

where

$$p(n) =$$

$$\exp\left[\frac{1}{2}\sigma\left(1-(n+1)\sin\left(\ln\left(n+1\right)\right)+\right.\right.$$
$$\left.\left.+\ (n+2)\sin\left(\ln\left(n+2\right)\right)\right)\right].$$

The sequence $p=(p(n)I_s)_{n\in\mathbb{N}}$ is a Lyapunov sequence and $p(n),\ n\in\mathbb{N}$ are positive, because

$$2\geq(n+2)\sin\ln\left(n+2\right)-(n+1)\sin\ln\left(n+1\right)\geq-2.$$

To obtain the last inequality it is enough to apply the Lagrange mean value theorem to the function $f:[n,n+1]\to\mathbb{R},\ f(x)=(x+1)\sin\ln(x+1)$. From the definition of the sequence $(p(n))_{n\in\mathbb{N}}$ we obtain that the transition matrix $\Phi_p\left(n,m\right)$ of (21) is given by

$$\Phi_p\left(n,m\right)=$$
$$I_s\exp\left[\frac{1}{2}\sigma\left((n+1)\sin\ln\left(n+1\right)+\right.\right.$$
$$\left.\left.-\ (m+1)\sin\ln\left(m+1\right)+n-m\right)\right].$$

From the last formula and Lemma 11 it follow that the Lyapunov spectrum of (21) is

$$\lambda\left(p\right)=(\sigma,...,\sigma)\in\mathbb{R}^s$$

and the Lyapunov spectrum $\lambda\left(p^{-T}\right)$ of the adjoint system is

$$\lambda\left(p^{-T}\right)=(0,...,0)\in\mathbb{R}^s.$$

Consequently, the Lyapunov regularity coefficient $\sigma_L\left(p\right)$ of system (21) is equal to

$$\sigma_L\left(p\right)=\sum_{i=1}^{s}\lambda_i\left(p\right)-\liminf_{n\to\infty}n^{-1}\ln\left|\det\Phi_p(n,0)\right|=s\sigma,$$

where to obtain the last equality we have used again Lemma 11. Since

$$\Phi=\left(I_s\exp\left[\frac{1}{2}\sigma\left((n+1)\sin\ln\left(n+1\right)+n\right)\right]\right)_{n\in\mathbb{N}}$$

is a normal fundamental matrix of (21), and the Lyapunov exponent $\lambda_i\left[\Phi^{-T}\right]$ of the ith row of Φ^{-T} is equal, by Lemma 11, to 0. This implies that the Grobman regularity coefficient $\sigma_G\left(p\right)$ is equal to

$$\sigma_G\left(p\right)=\max_{i=1,\ldots,s}\left\{\lambda_i\left(p\right)+\lambda_i\left[\Phi^{-T}\right]\right\}=\sigma.$$

By using Theorem 9, we construct a control U providing the dynamic equivalence of systems (7) and (21). Then the equalities (17) and (18) hold. The proof is completed.

The second main result of this paper is presented in the next theorem.

Theorem 12. If system (1) is uniformly completely controllable then for arbitrary numbers $\beta\geq\alpha$, there exists admissible feedback control $U=(U(n))_{n\in\mathbb{N}}$ such that for the closed loop system (7) we have

$$\Omega\left(A+BU\right)=\beta$$

and

$$\omega\left(A+BU\right)=\alpha.$$

Proof. We choose arbitrary numbers $\beta\geq\alpha$, set

$$p(n)=\exp\left[a(n+1)-a(n)\right],$$

where

$$a(n)=\frac{1}{2}n\alpha+\frac{1}{2}n\beta-\frac{1}{2}\alpha\sin\left(\ln\left(n+1\right)\right)+\frac{1}{2}\beta\sin\left(\ln\left(n+1\right)\right)$$

and consider system

$$y(n+1)=p(n)I_sy(n). \tag{22}$$

The transition matrix $\Phi_p\left(n,m\right)$ of (22) is given by

$$\Phi_p\left(n,m\right)=I_s\exp\left[a(n)-a(m)\right]$$

and therefore by Lemma 11 and (14) we have the following formulae for central exponents

$$\Omega\left(p\right)=\lim_{N\to\infty}\limsup_{n\to\infty}\frac{1}{nN}\sum_{i=1}^{n}\ln\left\|\Phi_p(iN,(i-1)\,N)\right\|=$$
$$\lim_{N\to\infty}\frac{1}{N}\limsup_{n\to\infty}\frac{1}{n}a\left(nN\right)=$$
$$\lim_{N\to\infty}\frac{1}{N}\limsup_{n\to\infty}\frac{1}{n}\left(\frac{1}{2}nN\alpha+\right.$$
$$+\frac{1}{2}nN\beta-\frac{1}{2}\alpha\sin\left(\ln\left(nN+1\right)\right)+$$
$$\left.+\frac{1}{2}\beta\sin\left(\ln\left(nN+1\right)\right)\right)=\beta,$$

$$\omega\left(A\right)=\lim_{N\to\infty}\liminf_{n\to\infty}\frac{1}{nN}\sum_{i=1}^{n}\ln\left\|\Phi^{-1}(iN,(i-1)\,N)\right\|^{-1}=$$
$$\lim_{N\to\infty}\frac{1}{N}\liminf_{n\to\infty}\frac{1}{n}a\left(nN\right)=\alpha.$$

Finally, using Theorem 9, we construct a control U providing the asymptotic equivalence of system (7) and (22).

4. CONCLUSIONS

In this paper we examined a problem of assignability by a feedback regularity coefficients and central exponents for discrete linear time-varying systems. Using the concept of scalarizability we showed that complete uniform controllability implies certain possibility of assignability of the values of Lyapunov, Perron and Grobman regularity coefficients and central exponents. An open question is possibility of arbitrary placement of regularity coefficients and central exponents as well as the question about simultaneous assignability of the Lyapunov spectrum, central exponents and regularity coefficients.

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