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A GENERALIZED α -WRIGHT CONVEXITY AND RELATED FUNCTIONAL EQUATION

JANUSZ MATKOWSKI AND MAŁGORZATA WRÓBEL

Abstract. Let I be an interval and $M, N : I \times I \rightarrow I$ some means with the strict internality property. Suppose that $\varphi : I \rightarrow \mathbb{R}$ is a non-constant and continuous solution of the functional equation

$$\varphi(M(x, y)) + \varphi(N(x, y)) = \varphi(x) + \varphi(y).$$

Then φ is one-to-one; moreover for every lower semicontinuous function $f : I \rightarrow \mathbb{R}$ satisfying the inequality

$$f(M(x, y)) + f(N(x, y)) \leq f(x) + f(y),$$

the function $f \circ \varphi^{-1}$ is convex on $\varphi(I)$. This is a generalization of an earlier result of Zs. Páles. An application to the α -Wright convex function is given.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $a \in (0, 1)$ a fixed number. A function $f : I \rightarrow \mathbb{R}$ is said to be α -Wright convex if, for all $x, y \in I$,

$$(1) \quad f(ax + (1-a)y) + f((1-a)x + ay) \leq f(x) + f(y).$$

It is shown in [3] that every lower semicontinuous α -Wright convex function is convex.

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Clearly, every linear function f converts (1) into equality. In this connection let us note that Zs. Páles [4] found a close relation between the more general functional inequality

$$(2) \quad f(M(x, y)) + f(N(x, y)) \leq f(x) + f(y), \quad x, y \in I,$$

and the corresponding functional equation

$$(3) \quad \varphi(M(x, y)) + \varphi(N(x, y)) = \varphi(x) + \varphi(y), \quad x, y \in I,$$

where $M, N : I \times I \rightarrow I$ are continuous functions satisfying the following strict internality condition

$$(4) \quad x, y \in I, x \neq y \Rightarrow M(x, y), N(x, y) \in (\min(x, y), \max(x, y)),$$

(in particular, M and N are means on I). He proved that: *if there exists a continuous strictly monotonic solution $\varphi : I \rightarrow \mathbb{R}$ of (3), then a continuous function $f : I \rightarrow \mathbb{R}$ satisfies (2) if, and only if, $f \circ \varphi^{-1}$ is a convex function on $\varphi(I)$* . In this note we show that this result remains true if φ is non-constant and continuous, and f lower semicontinuous.

2. Main result

The following result improves the result of Páles [4]

THEOREM. *Let $M, N : I \times I \rightarrow I$ be continuous functions satisfying condition (4), and suppose that $\varphi : I \rightarrow \mathbb{R}$ is a non-constant and continuous solution of equation (3). Then φ is one-to-one, and for every lower semicontinuous function $f : I \rightarrow \mathbb{R}$ satisfying inequality (2), the function $f \circ \varphi^{-1}$ is convex on $\varphi(I)$.*

PROOF. Put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Define $M_k, N_k : I \times I \rightarrow I$, $k \in \mathbb{N}_0$, by

$$M_0(x, y) := M(x, y), \quad N_0(x, y) := N(x, y),$$

$$M_{k+1}(x, y) := M(M_k(x, y), N_k(x, y)),$$

$$N_{k+1}(x, y) := N(M_k(x, y), N_k(x, y)),$$

and $m_k, n_k : I \times I \rightarrow I$, $k \in \mathbb{N}_0$,

$$m_k(x, y) := \min((M_k(x, y)), (N_k(x, y))),$$

$$n_k(x, y) := \max((M_k(x, y)), (N_k(x, y))).$$

Of course all the functions M_k, N_k, m_k, n_k are continuous. As M and N are means we have

$$\begin{aligned} m_0(x, y) \leq m_1(x, y) \leq \dots \leq m_k(x, y) \leq n_k(x, y) \\ \leq \dots \leq n_1(x, y) \leq n_0(x, y), \end{aligned}$$

and

$$(6) \quad M_k(x, y), N_k(x, y) \in [m_k(x, y), n_k(x, y)],$$

for all $k \in \mathbb{N}_0$ and $x, y \in I$. It follows that the sequences (m_k) and (n_k) converge on $I \times I$. Thus there exist $m_\infty, n_\infty : I \times I \rightarrow I$ such that

$$\lim_{k \rightarrow \infty} m_k(x, y) =: m_\infty(x, y) \leq n_\infty(x, y) := \lim_{k \rightarrow \infty} n_k(x, y),$$

for all $x, y \in I$. Since the functions of both sequences are continuous, (m_k) is increasing and (n_k) is decreasing, the function m_∞ is lower semicontinuous, and n_∞ is upper semicontinuous on $I \times I$. Suppose that there are $x, y \in I$ such that $m_\infty(x, y) < n_\infty(x, y)$. Hence, as M and N are the strict means, we would get

$$M(m_\infty(x, y), n_\infty(x, y)), N(m_\infty(x, y), n_\infty(x, y)) \in (m_\infty(x, y), n_\infty(x, y)).$$

Now the continuity of M and N implies that for sufficiently large k

$$M(M_k(x, y), N_k(x, y)), N(M_k(x, y), N_k(x, y)) \in (m_\infty(x, y), n_\infty(x, y)),$$

i.e.

$$M_{k+1}(x, y), N_{k+1}(x, y) \in (m_\infty(x, y), n_\infty(x, y)).$$

Hence, by the definition of the sequences (m_k) and (n_k) ,

$$m_{k+1}(x, y), n_{k+1}(x, y) \in (m_\infty(x, y), n_\infty(x, y)),$$

for sufficiently large k which is a contradiction. This proves that for all $x, y \in I$

$$m_\infty(x, y) = n_\infty(x, y).$$

Define $K : I \times I \rightarrow I$ by

$$K(x, y) := m_\infty(x, y), \quad x, y \in I.$$

The function K , being lower and upper semicontinuous, is continuous. The pointwise convergence of the sequences (M_k) and (N_k) to K is a consequence

of relation (6). Take $x, y \in I$ and $x \neq y$. Without any loss of generality we can assume that $x < y$. Then

$$x < M(x, y) < y, \quad x < N(x, y) < y.$$

Since

$$\min(M(x, y), N(x, y)) \leq K(x, y) \leq \max(M(x, y), N(x, y)),$$

we infer that K has strict internality property.

The definitions of $(M_k), (N_k)$, and relation (2) and (3), by an obvious induction imply, that for all $k \in \mathbb{N}$

$$f(M_k(x, y)) + f(N_k(x, y)) \leq f(x) + f(y), \quad x, y \in I,$$

and

$$\varphi(M_k(x, y)) + \varphi(N_k(x, y)) = \varphi(x) + \varphi(y), \quad x, y \in I.$$

Letting k tend to the infinity, and making use of the lower semicontinuity of f , the continuity of φ , and the relation

$$\lim_{k \rightarrow \infty} M_k(x, y) = K(x, y) = \lim_{k \rightarrow \infty} N_k(x, y),$$

which is a consequence of (6), we hence get

$$(7) \quad 2f(K(x, y)) \leq f(x) + f(y), \quad x, y \in I,$$

and

$$(8) \quad 2\varphi(K(x, y)) = \varphi(x) + \varphi(y), \quad x, y \in I.$$

Suppose that there are $a, b \in I$, $a \neq b$, such that $\varphi(a) = \varphi(b)$, and put

$$C := \{x \in I : \varphi(x) = \varphi(a)\}.$$

By the continuity of φ , the set C is closed in I . Note that C is an interval. In the opposite case we could find $a_1, b_1 \in C$, $a_1 < b_1$, such that $\varphi(x) \neq \varphi(a)$ for all $x \in [a_1, b_1]$. Setting in equation (8) $x = a_1, y = b_1$ we would get

$$2\varphi(K(a_1, b_1)) = \varphi(a_1) + \varphi(b_1) = 2\varphi(a),$$

i.e. $\varphi(K(a_1, b_1)) = \varphi(a)$, which according to the choice of the interval $[a_1, b_1]$ is impossible. Now the continuity of K and the property (6) easily imply that $C = I$. This contradiction proves that φ is one-to-one.

From (8) we have

$$K(x, y) = \varphi^{-1} \left(\frac{\varphi(x) + \varphi(y)}{2} \right), \quad x, y \in I.$$

Substituting this into (7) we get

$$2f \left[\varphi^{-1} \left(\frac{\varphi(x) + \varphi(y)}{2} \right) \right] \leq f(x) + f(y), \quad x, y \in I.$$

Setting here $x := \varphi^{-1}(s)$, $y := \varphi^{-1}(t)$, for $s, t \in \varphi(I)$ gives the Jensen convexity of the function $f \circ \varphi^{-1}$ on the interval $\varphi(I)$. This function is lower semicontinuous as the composition of the continuous function φ and lower semicontinuous function f . It follows that $f \circ \varphi^{-1}$ is convex (cf. for instance [1], Chapter I, Cor. 2.5). This completes the proof.

COROLLARY. *Let $I \subset \mathbb{R}$ be an interval and $a \in (0, 1)$ a fixed number. If $f : I \rightarrow \mathbb{R}$ is lower semicontinuous and*

$$f(ax + (1 - a)y) + f((1 - a)x + ay) \leq f(x) + f(y)$$

for all $x, y \in I$, then f is convex (and continuous).

PROOF. Since the function $\varphi : I \rightarrow \mathbb{R}$, $\varphi := \text{id}|_I$ is a non-constant and continuous solution of the functional equation

$$f(ax + (1 - a)y) + f((1 - a)x + ay) = f(x) + f(y), \quad x, y \in I,$$

and the functions $M, N : I \times I \rightarrow I$, defined by

$$M(x, y) := ax + (1 - a)y, \quad N(x, y) := (1 - a)x + ay, \quad x, y \in I,$$

are continuous means with the strict internality property, the result follows from the above theorem.

REMARK. The α -Wright convex functions appear in a natural way in connection with the converse of the Minkowski inequality (cf. [3]). Note that in [2] (answering to the question posed in [3]) it was shown that there exist α -Wright convex functions which are not Jensen convex.

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