REMARK ON INVARIANT STRAIGHT LINES OF SOME AFFINE TRANSFORMATIONS IN $\mathbb{R}^n$ WITHOUT FIXED POINTS

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Abstract. In this note we show that every affine transformation in the Euclidean space $\mathbb{R}^n$, which has no fixed points and fulfils the inequality $|f(x)f(y)| \leq |xy|$ for any x and y has invariant straight line.

In the book ([1], p. 203) is given, without proof, the following theorem: *every isometry in the Euclidean space* $\mathbb{R}^3$ *has an invariant straight line.* The same statement does not hold in $\mathbb{R}^n$, where $n \geq 2$ and $n \neq 3$.

In this note we generalize mentioned theorem for some affine transformations in $\mathbb{R}^n, n \geq 2$.

By $[ab]$ we will designate the straight line passing through the points a and b, and by $|ab|$ the distance between them.

We shall consider the affine transformations in $\mathbb{R}^n$ which satisfy the inequality

$$|f(x)f(y)| \leq |xy| \qquad \text{for any } x \text{ and } y. \tag{1}$$

They will be called the affine transformations which do not extend distances.

Theorem 1. *If f is an affine transformation in $\mathbb{R}^n$ without fixed points, then there exists a point x_0 that*

$$|x_0 f(x_0)| = \inf\{|xf(x)| : \ x \in \mathbb{R}^n\}. \tag{2}$$

Proof. Because $f(x) \neq x$ for any x, then the point $\Theta = (0, \dots, 0)$ does not belong to the range $g(\mathbb{R}^n)$ of the transformation g, where $g(x) = f(x) - x$. We conclude that the set $g(\mathbb{R}^n)$ is a k-dimensional hyperplane in $\mathbb{R}^n$, where $k < n$.

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The distance from the point Θ to this hyperplane is equal to the number $\inf\{|xf(x)| : x \in \mathbb{R}^n\}$.

Thus, there exists one point $p \in g(\mathbb{R}^n)$ such that $|\Theta p| = \inf\{|xf(x)| : x \in \mathbb{R}^n\}$. It is evident that there is a point $x_0 \in \mathbb{R}^n$ such that $f(x_0) - x_0 = p$, whence $|x_0 f(x_0)| = |\Theta p|$. □

THEOREM 2. *Let f be an affine transformation in $\mathbb{R}^n$ without fixed points which does not extend distances. If a and b be an arbitrary points satisfying the equality (2) then the following conditions hold:*

(i) the points $a, f(a), f(f(a))$ are different,

(ii) the points $a, f(a), f(f(a))$ are collinear,

(iii) the straight lines $[af(a)]$ and $[bf(b)]$ are parallel.

PROOF.

(i) The affine transformation f has no fixed points then $f(a) \neq a$ and $f(a) \neq f(f(a))$. We shall prove that $a \neq f(f(a))$. Indeed, if $a = f(f(a))$ then the midpoint of the segment $af(a)$ would be the fixed point of the transformation f. And this contradicts the assumption.

(ii) Let us assume that the points $a, f(a), f(f(a))$ are not collinear. Then the midpoint c of the segment $af(a)$ satisfies the equality $|cf(c)| = \frac{1}{2}|af(f(a))|$. In the triangle $a, f(a), f(f(a))$ is true the following inequality:

$$|cf(c)| < \frac{1}{2}(|af(a)| + |f(a)f(f(a))|).$$

Taking into account the inequality (1), i.e. $|f(a)f(f(a))| \leq |af(a)|$ we obtain the inequality $|cf(c)| < |af(a)|$, but this contradicts (2).

(iii) It follows from (ii) that the different straight lines $[af(a)]$ and $[bf(b)]$ remain unchanged under the affine transformation f which has no fixed points, then $[af(a)]$ and $[bf(b)]$ have no common point. They cannot be the skew lines. Let us assume the contrary. Then there exist a points $d \in [af(a)]$ and $e \in [b(f(b)]$ such that de is the unique shortest segment between the straight lines $[af(a)]$ and $[bf(b)]$. Since $f(d) \neq d$ and $f(e) \neq e$ thus $|f(d)f(e)| > |de|$, but this contradicts (1). □

The point p from the proof of the theorem 1 determines a hyperplane H in $\mathbb{R}^n$ by the formula:

$$H = \{x \in \mathbb{R}^n : \ f(x) - x = p\}.$$

It is easy to see that any point $x \in H$ satisfies the equality $|xf(x)| = |\Theta p|$ and conversely. The hyperplane H remains unchanged under the affine transformation f which does not extend distances and has no fixed points.

Indeed, if $a \in H$ then $|af(a)| \leq |f(a)f(f(a))|$. On the other hand $|f(a)f(f(a))| \leq |af(a)|$, because f does not extend distances. We conclude that $|f(a)f(f(a))| = |af(a)|$, i.e. $f(a) \in H$.

THEOREM 3. *If f is an affine transformation in $\mathbb{R}^n$ without fixed points and which does not extend distances then the restriction of the transformation f to the hyperplane H is a translation.*

PROOF. Let $a \in H$ be an arbitrary point. It follows from (ii) of the theorem 2 that the straight line $[af(a)]$ is invariant under the transformation f. The restriction of the transformation f to $[af(a)]$ is translation, because f has no fixed points.

If $\dim H = 1$ then the proof is finished.

Let us now assume that $\dim H > 1$, then there is $b \in H$ such that $[af(a)] \neq [bf(b)]$. It follows from (iii) of the theorem 2 that the straight lines $[af(a)]$ and $[bf(b)]$ are parallel.

Let $c \in [af(a)]$ and $d \in [bf(b)]$ be an arbitrary points such that the straight line $[cd]$ is perpendicular to the both straight lines $[af(a)]$ and $[bf(b)]$. The points $f(c)$ and $f(d)$ belong to $[af(a)]$ and $[bf(b)]$ respectively, then $|cd| \leq |f(c)f(d)|$. The transformation f does not extend distances then $|f(c)f(d)| \leq |cd|$. We obtain the equality $|cd| = |f(c)f(d)|$, i.e. the straight line $[f(c)f(d)]$ is perpendicular to the both straight lines $[af(a)]$ and $[bf(b)]$, and that what had to prove. $\square$

COROLLARY 1. *Every affine transformation in the Euclidean space $\mathbb{R}^n$ without fixed points which does not extend distances has invariant straight line.*

REMARK 1. If the affine transformation f extend distances, i.e.

$$\bigwedge_{x,y \in \mathbb{R}^n} |f(x)f(y)| \geq |xy|$$

and has no fixed points then the Corollary is true as well, because the inverse transformation f^{-1} of f does not extend distances and has no fixed points.

In the case when an affine transformation f is such that for some points x and y is $|f(x)f(y)| < |xy|$ and for another points u and v is $|uv| < |f(u)f(v)|$, the problem is open.

Reference

[1] Atanasjan L. S., Bazylew W. T., *Geometria, čast I,* Moskwa 1986.

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