FUNCTIONS OF CONVEXITY AND DIMENSION

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Abstract. Two dual sequence functions describing some kind of local convexity and dimension of subspaces of linear metric spaces are introduced. It is shown that the functions give a useful tool in the investigations of fixed point properties of the Schauder type.

Notations and conventions. By a linear metric space we mean a topological real vector space E which is metrizable. By Kakutani theorem (see for instance [6]) E is equipped with an *F-norm* such that $||x+y|| \leq ||x|| + ||y||$ and $||tx|| \leq ||x||$ for each $t \in [-1, 1]$. Such an F-norm induces an equivalent translation-invariant metric ρ on E given by the formula, $\rho(x, y) := ||x - y||$. A linear space with a metric induced by an F-norm is said to be an *F-metric* linear space. Let us denote by;
$B(a, r) := \{x \in E : \rho(x, a) < r\}$ — the ball with centre a and radius r,
$B(A, r) := \bigcup\{B(a, r) : a \in A\}$ for each nonempty set A,
$\operatorname{diam} A := \{\rho(x, y) : x, y \in E\}$ — the diameter of the set A,
$\operatorname{conv} A := \{x \in E : x = \sum_{i=0}^{n} t_i a_i,\ \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0,\ a_i \in A,\ n \in \mathbb{N}\}$ — the convex hull of the set A.

In this paper we want to construct a tool to estimate approximative fixed points. Our aim will be reach by constructing two dual sequences of functions describing dimension and local convexity.

Sequence function of dimension. For any family $\mathcal{W}$ of subsets of a metric space (Y, ρ) let us define *mesh* and *order* of the family $\mathcal{W}$:

mesh $\mathcal{W} < \varepsilon$ provided that diam $\mathcal{W} < \varepsilon$ for each $W \in \mathcal{W}$,
ord $\mathcal{W} \leq n$ provided that $|\{W \in \mathcal{W} : x \in W\}| \leq n + 1$ for each $x \in Y$.

Let us recall the definition of *covering dimension*, $\dim Y$, of a topological space

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Y; $\dim Y \le n$ provided that for each open finite covering $\mathcal{W}$ there exists an open finite covering $\mathcal{U}$ of order $\le n$, $\operatorname{ord}\mathcal{U} \le n$, being a refinement of $\mathcal{W}$ (i.e., for each $U \in \mathcal{U}$ there is $W \in \mathcal{W}$ such that $U \subset W$).

For a given metric space (Y, ρ) define a *sequence function of dimension* Ψ_Y: $\mathbb{N} \to [0, \infty)$:

$$\Psi_Y(n) := \inf\{\varepsilon > 0 : \exists \text{ finite covering } \mathcal{W} \text{ of } Y,\ \operatorname{mesh}\mathcal{W} < \varepsilon \ \text{ and } \operatorname{ord}\mathcal{W} \le n\}.$$

Let us list without proof the following properties of the function Ψ_Y:

1. $\Psi_Y(n) \ge \Psi_Y(n+1) \ge 0$ for each $n \in \mathbb{N}$.
2. If Y is a compact then $\lim_{n\to\infty} \Psi_Y(n) = 0$.
3. If $\dim Y < \infty$ and Y is compact then $\Psi_Y(n) = 0$ for each $n \ge \dim Y$.
4. $\Psi_Y(n) = \frac{1}{2^n}$ for the Hilbert cube $Y = [0,1]^\infty$, with the metric

$$\rho(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i}|x_i - y_i|.$$

THEOREM. *Let E be an infinite-dimensional F-metric linear space. Then for each decreasing sequence $\varepsilon_0 > \varepsilon_1 > \ldots > 0$ of reals there is a closed convex subset C of infinite dimension such that*

$$\Psi_C(n) < \varepsilon_n \quad \textit{for each } \ n \in \mathbb{N}.$$

PROOF. We shall define by induction a sequence of affine independent points $a_0, a_1, \ldots \in E$, a sequence of families $\mathcal{W}_n$, $n \in \mathbb{N}$, of open sets, and a sequence of positive reals $\delta_1 > \delta_2 > \ldots > 0$, $\delta_i < \varepsilon_i$ such that:

$$(1) \qquad \operatorname{mesh}\mathcal{W}_n < \varepsilon_n \ \textit{ and } \ \operatorname{ord}\mathcal{W}_n \le n \text{ for each } n \in \mathbb{N},$$

$$(2)\ C_n := \operatorname{conv}\{a_0, \ldots, a_n\} \subset \bigcup \mathcal{W}_n \subset B(C_{n-1}, \delta_n) \subset B(C_{n-1}, 2\delta_n) \subset \bigcup \mathcal{W}_{n-1}.$$

Inductive Construction.

Step 0. Choose $a_0 \in E \setminus \{0\}$ and define $C_0 := \{a_0\}$ and $\mathcal{W}_0 := \{E\}$.

Step $n+1$. Asume that we have defined affinely independent points $a_0, \ldots, a_n$ families $\mathcal{W}_0, \ldots, \mathcal{W}_n$ of open sets and reals $\delta_1, \ldots, \delta_n$ satisfying (1) and (2).

Since C_n is compact, there exists a positive real δ_{n+1}; $0 < \delta_{n+1} < \delta_n$, $2\delta_{n+1} \le \varepsilon_{n+1}$, such that

$$C_n \subset B(C_n, \delta_{n+1}) \subset B(C_n, 2\delta_{n+1}) \subset \bigcup \mathcal{W}_n$$

Choose a point $a_{n+1} \in B(C_n, \delta_{n+1}) \setminus \operatorname{span} C_n$. The points $a_0, \ldots, a_{n+1}$ are affinely independent. Note that

$$C_{n+1} := \operatorname{conv}\{a_0, \ldots, a_{n+1}\} \subset B(C_n, \delta_{n+1}).$$

To see this, fix $x \in C_{n+1}$. Then

$$x = \sum_{i=0}^{n+1} t_i a_i, \quad \sum_{i=0}^{n+1} t_i = 1 \text{ and } t_i \geq 0.$$

Choose $b \in C_n$ such that $||a_{n+1} - b|| < \delta_{n+1}$ and put

$$y := \sum_{i=0}^{n} t_i a_i + t_{n+1} b.$$

Then it is clear that $y \in C_n$ and

$$||x - y|| = ||t_{n+1}(a_{n+1} - b)|| \leq ||a_{n+1} - b|| < \delta_{n+1}.$$

This yields $x \in B(C_n, \delta_{n+1})$. Since dim $C_{n+1} = n + 1$, according to theorems on shrinkings and swellings of families of sets (see [1], Theorems 1.7.8 and 3.1.2), one can find a family $\mathcal{W}_{n+1}$ of open sets in E such that

$$\operatorname{mesh} \mathcal{W}_{n+1} < \varepsilon_{n+1}, \ \operatorname{ord} \mathcal{W}_{n+1} \leq n+1, \ C_{n+1} \subset \bigcup \mathcal{W}_{n+1} \subset B(C_n, \delta_{n+1}).$$

This completes the inductive construction. Now, let us put

$$C := \overline{\bigcup_{n=0}^{\infty} C_n}.$$

Note that

$$C \subset \bigcap_{i=0}^{\infty} \overline{B(C_n, \delta_{n+1})},$$

because $\bigcup_{n=0}^{\infty} C_n \subset \bigcap_{n=0}^{\infty} \overline{B(C_n, \delta_{n+1})}$. Thus from (1) and (2) we infer that $C \subset \bigcup \mathcal{W}_n$ for each $n \in \mathbb{N}$, and therefore $\Psi_C(n) \leq \operatorname{mesh} \mathcal{W}_n < \varepsilon_n$. □

Sequence function of convexity. For a given subset $Y \subset E$ of a linear metric space (E, ρ) define a *sequence function of convexity* $\Phi_Y : \mathbb{N} \times [0, \infty) \to [0, \infty)$;

$\Phi_Y(n, r) := \inf\{L > r : \forall_{K>L} \exists_{s>r} \forall_{y, c_0, \ldots, c_n \in Y} \ c_0, \ldots, c_n \in B(y, s) \Longrightarrow \operatorname{conv}\{c_0, \ldots, c_n\} \subset B(y, K)\}$.

The function Φ_Y has the following properties:

1. $\Phi_Y(n, r) \leq \Phi_Y(n+1, r)$ and $\Phi(n, r) \leq \Phi(n, s)$ for each $n \in \mathbb{N}$ and $r \leq s$.
2. $\Phi_Z(n, r) \leq \Phi_Y(n, r)$ for $Z \subset Y$.
3. If $(E, ||\cdot||)$ is a normed space, then $\Phi_Y(n, r) = r$ for each $n \in \mathbb{N}$ and $r \geq 0$.
4. If (E, ρ) is an F-metric linear space, then $\Phi_Y(n, r) \leq (n+1)r$.

To see this, let $c_0, \ldots, c_n \in B(y, s)$. Choose $x \in \operatorname{conv}\{c_0, \ldots, c_n\} \subset B(y, s)$. Then

$$\rho(x, y) = ||\sum_{i=0}^{n} t_i c_i - y|| = ||\sum_{i=0}^{n} t_i c_i - \sum_{i=0}^{n} t_i y||$$

$$\leq \sum_{i=0}^{n} ||t_i(c_i - y)|| \leq \sum_{i=0}^{n} ||c_i - y|| \leq (n+1)s,$$

where

$$\sum_{i=0}^{n} t_i = 1, \quad t_i \geq 0, \quad K > (n+1)r, \quad r < s < \frac{K}{n+1}.$$

5. Fix $0 < p < 1$. Recall that the Lebesgue space L_p is defined to be an F-metric space of all the Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$ with an F-norm such that

$$||f|| := \int_0^1 |f(t)|^p \, dt \quad < \infty.$$

One can verify that $\Phi_Y(n,r) \leq r(n+1)^{1-p}$.

Raughly speaking, a function of convexity Φ_Y describes some kind of n-local convexity of nonlocally convex F-metric spaces. This function together with a sequence function of dimension Ψ_Y gives a better tool for investigations of a fixed point property, than a sequence function of the Kuratowski measure of noncompactness [5]. Some methods of measure of noncompactness which are intensively exploit the reader will find in [7].

6. In a paper [4] due to Olga Hadžić it is investigated a notion of a set of Z_ϕ-type. In our terminology a subset $Y \subset E$ of an F-metric linear space E is said to be of Z_ϕ-type if there exists a function $\phi : [0,\infty) \to [0,\infty)$ such that for each $r > 0$

$$\operatorname{conv}[(Y - Y) \cap B(0,r)] \subset B(0,\phi(r)).$$

From this condition it follows that

$$\Phi_Y(n,r) \leq \phi(r) \quad \text{for each } n \in \mathbb{N}, \quad r > 0.$$

7. In the same paper, for the Lebegue space L_0;

$$L_0 := \{f : [0,1] \to \mathbb{R} \ : \ ||f|| = \int_0^1 \frac{f(t)}{1+f(t)} dt < \infty\},$$

it is shown that for the convex set

$$Y_A := \{f \in L_0 : |f(t)| \leq A \text{ for each } t \in [0,1]\}, \text{ where } A > 0,$$

the function ϕ is given by the formula:

$$\phi(r) = (1+2A)r.$$

The concept of Z_ϕ-set was originated in Zima's paper [8], where a fixed point property of the Schauder type was established for some nonlocally F-metric spaces. From the results of the next part of our paper it will be follow that for this space Y_A each continuous compact map $g : Y_A \to Y_A$ has a fixed point.

Mixed sequence of functions of convexity and dimension. A function $\chi_Y : \mathbb{N} \to [0, \infty)$, where Y is a subset of a linear metric space E, defined by the formula

$$\chi_Y(n) := \Phi_Y[n, \Psi_Y(n)]$$

is said to be a *mixed sequence function of convexity and dimension.* The real number

$$\chi(Y) := \inf\{\chi_Y(n) : n \in \mathbb{N}\}$$

is said to the *convexity-dimension characteristic* of the subset Y of E.

The following properties of the function χ are easy to deduce.

1. If (E, ρ) is an F-metric linear space, then $\chi_Y(n) \leq (n+1)\Psi_Y(n)$ for each $n \in N$.
2. If E is a normed space, then $\chi_Y(n) = \Psi_Y(n)$, for each subset $Y \subset E$, and consequently:
3. If Y is a subset of a normed space E, then $\chi(Y) = 0$.
4. If Y is a compact subset of an F-metric space E and dim $Y < \infty$, then $\chi(Y) = 0$.
5. Let Y be a set of Z_ϕ-type in an F-metric space E. Then $\chi_Y(n) \leq \phi(\chi_Y(n))$ and $\chi(Y) \leq \lim_{n\to\infty} \phi(\Psi_Y(n))$.
6. For each subset $Y \subset L_p$, if $0 \leq p < 1$ then $\chi_Y(n) \leq (n+1)^{1-p}\Psi_Y(n)$.

Now, we are going to show some applications in investigating of a fixed point property of the Schauder type.

MAIN THEOREM. *Let $Y \subset X \subset E$ be an arbitrary subset of a convex set X of a linear metric space E. Fix $n \in N$ and $K > \chi_Y(n)$. Then for each continuous map $g : X \to Y$ there is a point $c \in X$ such that $\rho(c, g(c)) < K$.*

PROOF. By definition $K > \chi(n)$ means that

$$\Phi_Y[n, \Psi_Y(n)] < K, \tag{1}$$

and let us put

$$r := \Psi_Y(n) \quad \text{and} \quad L := \Phi_Y(n, r) \tag{2}$$

According to the definitions of functions Φ_Y there is $s > r$ such that for each $y, c_0, \ldots, c_n \in Y$

$$c_0, \ldots, c_n \in B(y, s) \Longrightarrow \operatorname{conv}\{c_0, \ldots, c_n\} \subset B(y, K). \tag{3}$$

Now, from the definition of the function Ψ_Y there exists a finite relatively open covering $\mathcal{W} = \{W_0, \ldots, W_m\}$ of Y such that

$$\operatorname{ord} \mathcal{W} \leq n \quad \text{and} \quad \operatorname{mesh} \mathcal{W} < s. \tag{4}$$

Choose points $c_i \in W_i$ for each $i = 0, \dots, m$.

We shall show that there exists a point $c \in X$ and a sequence of indices $0 \leq i_0 < \dots < i_k \leq m$ such that

$$c \in \operatorname{conv}\{c_{i_0}, \dots, c_{i_k}\} \cap g^{-1}(W_{i_0}) \cap \dots \cap g^{-1}(W_{i_k}). \tag{5}$$

Indeed, if not, then $\operatorname{conv}\{i_0, \dots, i_k\} \subset F_{i_0} \cup \dots \cup F_{i_k}$ for each set $0 \leq i_0 < \dots < i_k \leq m$ of indices, where $F_i = X \setminus g^{-1}(W_i)$. Then according to the KKM-principle (see [2] , Theorem 1.2, p.73 or [3] Theorem 8.2, p.97) the intersection $\bigcap\{F_i : i = 1, \dots, m\}$ is a nonempty set. This contradicts the fact that the family $\{g^{-1}(W_i) : i =, \dots, m\}$ is a covering of X.

From (4–5) and $c_i \in W_i$ it follows that

$$k \leq n \quad \text{and} \quad c_{i_0}, \dots, c_{i_k} \in B(g(c), s). \tag{6}$$

From (3) we get

$$c \in \operatorname{conv}\{c_{i_0}, \dots, c_{i_k}\} \subset B(g(c), K). \tag{7}$$

Finally, we have obtained $\rho(c, g(c)) < K$. □

THEOREM. *Let $Y \subset X \subset E$ be a compact subset of a convex set X of a linear metric space E such that $\chi(Y) = 0$. Then every continuous map $g : X \to Y$ has a fixed point.*

PROOF. According to Main Theorem for each $\varepsilon > 0$ there exists a point $c_\varepsilon \in X$ such that $\rho(g(c_\varepsilon), c_\varepsilon) < \varepsilon$. Using compatness arguments we may assume that there is a point $c \in X$ such that $c_\varepsilon \longrightarrow c$ as $\varepsilon \longrightarrow 0$. The continuity of g yields $g(c) = c$. □

If we assume that balls $B(x, r)$ are convex then it is clear that $\Psi_Y(n, r) = r$ for each $n \in N$ and $r > 0$ and consequently $\chi(Y) = 0$ for each compact subspace of E. Thus, we immediately obtain:

COROLLARY 1. (The Schauder fixed point theorem). *Let X be a convex subset of a metric linear space E such that open balls are convex. Then each continuous map $g : X \to X$, where $\overline{g(X)}$ is compact, has a fixed point.*

From the properties of the function χ we also obtain

COROLLARY 2. *Let $Y \subset X \subset E$ be a compact subset of a convex subset of an F-metric space E. If dim $Y < \infty$, then each continuous map $g : X \to Y$ has a fixed point.*

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