Title: Functions of convexity and dimension

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FUNCTIONS OF CONVEXITY AND DIMENSION

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Abstract. Two dual sequence functions describing some kind of local convexity and dimension of subspaces of linear metric spaces are introduced. It is shown that the functions give a useful tool in the investigations of fixed point properties of the Schauder type.

Notations and conventions. By a linear metric space we mean a topological real vector space $E$ which is metrizable. By Kakutani theorem (see for instance [6]) $E$ is equipped with an $F$-norm such that $||x+y|| \leq ||x|| + ||y||$ and $||tx|| \leq ||x||$ for each $t \in [-1,1]$. Such an $F$-norm induces an equivalent translation-invariant metric $\rho$ on $E$ given by the formula, $\rho(x,y) := ||x - y||$. A linear space with a metric induced by an $F$-norm is said to be an $F$-metric linear space. Let us denote by;

$B(a,r) := \{x \in E : \rho(x,a) < r\}$ — the ball with centre $a$ and radius $r$,
$B(A,r) := \bigcup\{B(a,r) : a \in A\}$ for each nonempty set $A$,
$diam A := \{\rho(x,y) : x,y \in E\}$ — the diameter of the set $A$,
$conv A := \{x \in E : x = \sum_{i=0}^{n} t_i a_i, \sum_{i=0}^{n} t_i = 1, t_i \geq 0, a_i \in A, n \in \mathbb{N}\}$ — the convex hull of the set $A$.

In this paper we want to construct a tool to estimate approximative fixed points. Our aim will be reach by constructing two dual sequences of functions describing dimension and local convexity.

Sequence function of dimension. For any family $\mathcal{W}$ of subsets of a metric space $(Y, \rho)$ let us define mesh and order of the family $\mathcal{W}$:

mesh $\mathcal{W} < \varepsilon$ provided that $diam \mathcal{W} < \varepsilon$ for each $W \in \mathcal{W}$,
ord $\mathcal{W} \leq n$ provided that $|\{W \in \mathcal{W} : x \in W\}| \leq n + 1$ for each $x \in Y$.

Let us recall the definition of covering dimension, $\dim Y$, of a topological space
For a given metric space \((Y, \rho)\) define a sequence function of dimension \(\Psi_Y\): 
\[\Psi_Y(n) := \inf \{ \varepsilon > 0 : \exists \text{ finite covering } W \text{ of } Y, \text{ mesh } W < \varepsilon \text{ and ord } W \leq n \}.\]

Let us list without proof the following properties of the function \(\Psi_Y\):

1. \(\Psi_Y(n) \geq \Psi_Y(n + 1) > 0\) for each \(n \in \mathbb{N}\).
2. If \(Y\) is a compact then \(\lim_{n \to \infty} \Psi_Y(n) = 0\).
3. If \(\dim Y < \infty\) and \(Y\) is compact then \(\Psi_Y(n) = 0\) for each \(n \geq \dim Y\).
4. \(\Psi_Y(n) = \frac{1}{2^n}\) for the Hilbert cube \(Y = [0,1]^\infty\), with the metric
\[\rho(x,y) := \sum_{i=1}^{\infty} \frac{1}{2^i} |x_i - y_i|\]

**Theorem.** Let \(E\) be an infinite-dimensional F-metric linear space. Then for each decreasing sequence \(\varepsilon_0 > \varepsilon_1 > \ldots > 0\) of reals there is a closed convex subset \(C\) of infinite dimension such that
\[\Psi_C(n) < \varepsilon_n \text{ for each } n \in \mathbb{N}.\]

**Proof.** We shall define by induction a sequence of affine independent points \(a_0, a_1, \ldots \in E\), a sequence of families \(W_n, n \in \mathbb{N}\), of open sets, and a sequence of positive reals \(\delta_1 > \delta_2 > \ldots > 0, \delta_i < \varepsilon_i\) such that:

1. \(\text{mesh } W_n < \varepsilon_n\) and \(\text{ord } W_n \leq n\) for each \(n \in \mathbb{N}\),
2. \(C_n := \text{conv } \{a_0, \ldots, a_n\} \subset \bigcup W_n \subset B(C_{n-1}, \delta_n) \subset B(C_{n-1}, 2\delta_n) \subset \bigcup W_{n-1}\).

**Inductive Construction.**

*Step 0.* Choose \(a_0 \in E \setminus \{0\}\) and define \(C_0 := \{a_0\}\) and \(W_0 := \{E\}\).

*Step \(n + 1\).* Assume that we have defined affinely independent points \(a_0, \ldots, a_n\) families \(W_0, \ldots, W_n\) of open sets and reals \(\delta_1, \ldots, \delta_n\) satisfying (1) and (2).

Since \(C_n\) is compact, there exists a positive real \(\delta_{n+1}; 0 < \delta_{n+1} < \delta_n, 2\delta_{n+1} \leq \varepsilon_{n+1}\), such that
\[C_n \subset B(C_n, \delta_{n+1}) \subset B(C_n, 2\delta_{n+1}) \subset \bigcup W_n\]

Choose a point \(a_{n+1} \in B(C_n, \delta_{n+1}) \setminus \text{span } C_n\). The points \(a_0, \ldots, a_{n+1}\) are affinely independent. Note that
\[C_{n+1} := \text{conv } \{a_0, \ldots, a_{n+1}\} \subset B(C_n, \delta_{n+1}).\]
To see this, fix $x \in C_{n+1}$. Then
\[ x = \sum_{i=0}^{n+1} t_i a_i, \quad \sum_{i=0}^{n+1} t_i = 1 \text{ and } t_i \geq 0. \]
Choose $b \in C_n$ such that $||a_{n+1} - b|| < \delta_{n+1}$ and put
\[ y := \sum_{i=0}^{n} t_i a_i + t_{n+1} b. \]

Then it is clear that $y \in C_n$ and
\[ ||x - y|| = ||t_{n+1}(a_{n+1} - b)|| \leq ||a_{n+1} - b|| < \delta_{n+1}. \]
This yields $x \in B(C_n, \delta_{n+1})$. Since $\dim C_{n+1} = n + 1$, according to theorems on shrinkings and swellings of families of sets (see [1], Theorems 1.7.8 and 3.1.2), one can find a family $\mathcal{W}_{n+1}$ of open sets in $E$ such that
\[ \text{mesh } \mathcal{W}_{n+1} < \varepsilon_{n+1}, \quad \text{ord } \mathcal{W}_{n+1} \leq n + 1, \quad C_{n+1} \subset \bigcup \mathcal{W}_{n+1} \subset B(C_n, \delta_{n+1}). \]
This completes the inductive construction. Now, let us put
\[ C := \bigcup_{n=0}^{\infty} C_n. \]

Note that
\[ C \subset \bigcap_{n=0}^{\infty} B(C_n, \delta_{n+1}), \]
because $\bigcup_{n=0}^{\infty} C_n \subset \bigcap_{n=0}^{\infty} B(C_n, \delta_{n+1})$. Thus from (1) and (2) we infer that $C \subset \bigcup \mathcal{W}_n$ for each $n \in \mathbb{N}$, and therefore $\Psi_C(n) \leq \text{mesh } \mathcal{W}_n < \varepsilon_n$. □

**Sequence function of convexity.** For a given subset $Y \subset E$ of a linear metric space $(E, \rho)$ define a sequence function of convexity $\Phi_Y : \mathbb{N} \times [0, \infty) \to [0, \infty)$;
\[ \Phi_Y(n, r) := \inf\{ L > r : \forall K > L \exists s > r \forall y, c_0, \ldots, c_n \in Y \quad c_0, \ldots, c_n \in B(y, s) \Rightarrow \conv\{c_0, \ldots, c_n\} \subset B(y, K)\}. \]

The function $\Phi_Y$ has the following properties:
1. $\Phi_Y(n, r) \leq \Phi_Y(n + 1, r)$ and $\Phi(n, r) \leq \Phi(n, s)$ for each $n \in \mathbb{N}$ and $r \leq s$.
2. $\Phi_Z(n, r) \leq \Phi_Y(n, r)$ for $Z \subset Y$.
3. If $(E, || \cdot ||)$ is a normed space, then $\Phi_Y(n, r) = r$ for each $n \in \mathbb{N}$ and $r \geq 0$.
4. If $(E, \rho)$ is an F-metric linear space, then $\Phi_Y(n, r) \leq (n + 1)r$.

To see this, let $c_0, \ldots, c_n \in B(y, s)$. Choose $x \in \conv\{c_0, \ldots, c_n\} \subset B(y, s)$. Then
\[ \rho(x, y) = ||\sum_{i=0}^{n} t_i c_i - y|| = ||\sum_{i=0}^{n} t_i c_i - \sum_{i=0}^{n} t_i y||. \]
\[ \leq \sum_{i=0}^{n} ||t_i(c_i - y)|| \leq \sum_{i=0}^{n} ||c_i - y|| \leq (n+1)s, \]

where
\[ \sum_{i=0}^{n} t_i = 1, \quad t_i \geq 0, \quad K > (n+1)r, \quad r < s < \frac{K}{n+1}. \]

5. Fix \( 0 < p < 1. \) Recall that the Lebesgue space \( L_p \) is defined to be an F-metric space of all the Lebesgue measurable functions \( f : [0,1] \to \mathbb{R} \) with an F-norm such that
\[ ||f|| := \int_{0}^{1} |f(t)|^p \, dt < \infty. \]

One can verify that \( \Phi_Y(n,r) \leq r(n+1)^{1-p}. \)

Roughly speaking, a function of convexity \( \Phi_Y \) describes some kind of \( n \)-local convexity of nonlocally convex F-metric spaces. This function together with a sequence function of dimension \( \Psi_Y \) gives a better tool for investigations of a fixed point property, than a sequence function of the Kuratowski measure of noncompactness \( [5] \). Some methods of measure of noncompactness which are intensively exploit the reader will find in \( [7] \).

6. In a paper \( [4] \) due to Olga Hadžić it is investigated a notion of a set of \( Z_\phi \)-type. In our terminology a subset \( Y \subset E \) of an F-metric linear space \( E \) is said to be of \( Z_\phi \)-type if there exists a function \( \phi : [0,\infty) \to [0,\infty) \) such that for each \( r > 0 \)
\[ \text{conv} ([Y - Y] \cap B(0,r)) \subset B(0,\phi(r)). \]

From this condition it follows that
\[ \Phi_Y(n,r) \leq \phi(r) \quad \text{for each } n \in \mathbb{N}, \quad r > 0. \]

7. In the same paper, for the Lebesgue space \( L_0; \)
\[ L_0 := \{ f : [0,1] \to \mathbb{R} : ||f|| = \int_{0}^{1} \frac{f(t)}{1+f(t)} \, dt < \infty \}, \]
it is shown that for the convex set
\[ Y_A := \{ f \in L_0 : |f(t)| \leq A \text{ for each } t \in [0,1] \}, \text{ where } A > 0, \]
the function \( \phi \) is given by the formula:
\[ \phi(r) = (1 + 2A)r. \]

The concept of \( Z_\phi \)-set was originated in Zima's paper \( [8] \), where a fixed point property of the Schauder type was established for some nonlocally F-metric spaces. From the results of the next part of our paper it will be follow that for this space \( Y_A \) each continuous compact map \( g : Y_A \to Y_A \) has a fixed point.
Mixed sequence of functions of convexity and dimension. A function \( \chi_Y : \mathbb{N} \to [0, \infty) \), where \( Y \) is a subset of a linear metric space \( E \), defined by the formula

\[
\chi_Y(n) := \Phi_Y[n, \Psi_Y(n)]
\]
is said to be a mixed sequence function of convexity and dimension. The real number

\[
\chi(Y) := \inf\{\chi_Y(n) : n \in \mathbb{N}\}
\]
is said to be the convexity-dimension characteristic of the subset \( Y \) of \( E \).

The following properties of the function \( \chi \) are easy to deduce.

1. If \((E, \rho)\) is an F-metric linear space, then \( \chi_Y(n) \leq (n + 1)\Psi_Y(n) \) for each \( n \in \mathbb{N} \).

2. If \( E \) is a normed space, then \( \chi_Y(n) = \Psi_Y(n) \), for each subset \( Y \subset E \), and consequently:

3. If \( Y \) is a subset of a normed space \( E \), then \( \chi(Y) = 0 \).

4. If \( Y \) is a compact subset of an F-metric space \( E \) and \( \dim Y < \infty \), then \( \chi(Y) = 0 \).

5. Let \( Y \) be a set of \( \Phi \)-type in an F-metric space \( E \). Then \( \chi_Y(n) \leq \phi(\chi_Y(n)) \) and \( \chi(Y) \leq \lim_{n \to \infty} \phi(\Psi_Y(n)) \).

6. For each subset \( Y \subset L_p \), if \( 0 \leq p < 1 \) then \( \chi_Y(n) \leq (n + 1)^{1-p}\Psi_Y(n) \).

Now, we are going to show some applications in investigating of a fixed point property of the Schauder type.

**MAIN THEOREM.** Let \( Y \subset X \subset E \) be an arbitrary subset of a convex set \( X \) of a linear metric space \( E \). Fix \( n \in \mathbb{N} \) and \( K > \chi_Y(n) \). Then for each continuous map \( g : X \to Y \) there is a point \( c \in X \) such that \( \rho(c, g(c)) < K \).

**PROOF.** By definition \( K > \chi(n) \) means that

(1) \[ \Phi_Y[n, \Psi_Y(n)] < K, \]

and let us put

(2) \[ r := \Psi_Y(n) \text{ and } L := \Phi_Y(n, r) \]

According to the definitions of functions \( \Phi_Y \) there is \( s > r \) such that for each \( y, c_0, \ldots, c_n \in Y \)

(3) \[ c_0, \ldots, c_n \in B(y, s) \implies \text{conv} \{c_0, \ldots, c_n\} \subset B(y, K). \]

Now, from the definition of the function \( \Psi_Y \) there exists a finite relatively open covering \( \mathcal{W} = \{W_0, \ldots, W_m\} \) of \( Y \) such that

(4) \[ \text{ord} \mathcal{W} \leq n \text{ and } \text{mesh} \mathcal{W} < s. \]
Choose points $c_i \in W_i$ for each $i = 0, \ldots, m$.

We shall show that there exists a point $c \in X$ and a sequence of indices $0 \leq i_0 < \ldots < i_k \leq m$ such that

$$(5) \quad c \in \text{conv} \{c_{i_0}, \ldots, c_{i_k}\} \cap g^{-1}(W_{i_0}) \cap \ldots \cap g^{-1}(W_{i_k}).$$

Indeed, if not, then $\text{conv} \{i_0, \ldots, i_k\} \subset F_{i_0} \cup \ldots \cup F_{i_k}$ for each set $0 \leq i_0 < \ldots < i_k \leq m$ of indices, where $F_i = X \setminus g^{-1}(W_i)$. Then according to the KKM-principle (see [2], Theorem 1.2, p.73 or [3] Theorem 8.2, p.97) the intersection $\bigcap \{F_i : i = 1, \ldots, m\}$ is a nonempty set. This contradicts the fact that the family $\{g^{-1}(W_i) : i = 1, \ldots, m\}$ is a covering of $X$.

From (4–5) and $c_i \in W_i$ it follows that

$$(6) \quad k \leq n \quad \text{and} \quad c_{i_0}, \ldots, c_{i_k} \in B(g(c), s).$$

From (3) we get

$$(7) \quad c \in \text{conv} \{c_{i_0}, \ldots, c_{i_k}\} \subset B(g(c), K).$$

Finally, we have obtained $\rho(c, g(c)) < K$. \hfill \square

**Theorem.** Let $Y \subset X \subset E$ be a compact subset of a convex set $X$ of a linear metric space $E$ such that $x(Y) = 0$. Then every continuous map $g : X \rightarrow Y$ has a fixed point.

**Proof.** According to Main Theorem for each $\varepsilon > 0$ there exists a point $c_\varepsilon \in X$ such that $\rho(g(c_\varepsilon), c_\varepsilon) < \varepsilon$. Using compactness arguments we may assume that there is a point $c \in X$ such that $c_\varepsilon \rightarrow c$ as $\varepsilon \rightarrow 0$. The continuity of $g$ yields $g(c) = c$. \hfill \square

If we assume that balls $B(x, r)$ are convex then it is clear that $\Psi_Y(n, r) = r$ for each $n \in N$ and $r > 0$ and consequently $x(Y) = 0$ for each compact subspace of $E$. Thus, we immediately obtain:

**Corollary 1.** (The Schauder fixed point theorem). Let $X$ be a convex subset of a metric linear space $E$ such that open balls are convex. Then each continuous map $g : X \rightarrow X$, where $g(X)$ is compact, has a fixed point.

From the properties of the function $x$ we also obtain

**Corollary 2.** Let $Y \subset X \subset E$ be a compact subset of a convex subset of an $F$-metric space $E$. If $\dim Y < \infty$, then each continuous map $g : X \rightarrow Y$ has a fixed point.
References


