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Author: Władysław Kulpa, Andrzej Szymański

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APPLICATIONS OF AUTOMATIC QUASI-CONTINUITY

WŁADYSŁAW KULPA, ANDRZEJ SZYMAŃSKI

Abstract. We study quasi-continuous functions on the product of two spaces provided they are separately continuous. We apply our results to actions of (semi-) groups on topological spaces and to the problem of the uniqueness of extensions of separately continuous functions.

1. Introduction

It was probably R. Baire [2] who first distinguished functions defined on the product $\mathbf{R}\times\mathbf{R}$ which are continuous after making one of the two variables constant. For our purposes, we need a slightly more general notion. We say that a function $f : X_1 \times ... \times X_n \to Z$ defined on the Cartesian product of topological spaces $X_1, ..., X_n$ into space Z is *separately (quasi-)continuous* if for each $1 \leq k \leq n$ and $a_k \in X_k$ the function $f(a_1, .., a_{k-1}, \cdot, a_{k+1}, .., a_n) : X_k \to Z$ is (quasi-)continuous.

Over the years, many studies have appeared which describe continuity-like properties of (separately continuous) functions. One of such properties is *quasi-continuity.* Originated by S. Kempisty [10], it has been studied quite intensely afterwards (see, for example, [1], [3]). A comprehensive survey of quasi-continuous functions is given in [14].

In our paper, we use a quite general and non-traditional approach to quasi-continuous functions, which allows us to place them among such functions as cliquish functions or barely continuous ones (see Section 3).

The motivations for studying quasi-continuous functions in general setting come from two directions. One is due to W. Sierpiński [17] and is related to the uniqueness of extensions of separately continuous functions. The other one pertains determining whether a separately continuous action of a semi-group on a topological space

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is actually continuous. By noticing that in some instances separately continuous functions are quasi-continuous, the so called *automatic quasi-continuity*, we were able to get positive answers to the aforementioned problems in some cases (cf. Corollary 3 and Theorem 3).

For all undefined (topological or settheoretical) notions we refer to [6] and [9].

2. Quasi-continuous and other functions

For an arbitrary function $f : X \to Y$ and an open cover $\mathcal{P}$ of the space Y, let

$$\omega(f;\mathcal{P}) = \{x \in X : \exists_{U\ open}\exists_{V\in\mathcal{P}}\ (x \in U\ and\ f(U) \subseteq V)\}.$$

If M is a metric space and $\mathcal{P}_\varepsilon$ is the family of all open balls of diameter less than ε, then $\omega(f;\mathrm{P}_\varepsilon)$ is the set of all points where the oscillation of f does not exceed ε.

Sets of the form $\omega(f;\mathcal{P})$ are open. Let us call a function f to be $\mathcal{P}$ *– cliquish* if the set $\omega(f;\mathcal{P})$ is also dense in the space X.

A sequence $\{\mathcal{P}_n : n = 1,2,...\}$ of open covers of a space Y is said to be a *development* for Y if for each $y \in Y$ and for each selection U_n from $\mathcal{P}_n$, each containing the point y, the family $\{U_n : n = 1,2,...\}$ is a base at the point y.

A subset F of a topological space is said to be *nowhere dense* if the interior of its closure is empty, i.e., $Int\,(clF) = \emptyset$. A countable union of nowhere dense sets is called a *meager set.* The complement of a meager set is called a *residual set.*

For arbitrary topological space X, the *Baire number,* $b(X)$, is defined as follows:

$$b\,(X) = \inf\left\{|\mathcal{R}| :\ \ \forall_{F\in\mathcal{R}}\ F \text{ is nowhere dense and } Int\left(\bigcup\mathcal{R}\right) \neq \emptyset\right\}.$$

Spaces with an uncountable Baire number are called *Baire spaces.*

LEMMA 1. *Let $\{\mathcal{P}_n : n = 1,2,...\}$ be a development for Y and let $f : X \to Y$ be a $\mathcal{P}_n$ – cliquish function, for each $n = 1,2,...$ Then the function f is continuous at each point of a residual subset of X.*

PROOF. The set in question is the intersection of all sets $\omega(f;\mathcal{P}_n)$. Indeed, if $x \in \omega(f;\mathcal{P}_n)$ for each $n = 1,2,...$, then one can find $V_n \in \mathcal{P}_n$ and open neighborhoods U_n of x satisfying $f(U_n) \subseteq V_n$, $n = 1,2,...$ Hence $\{V_n : n = 1,2,...\}$ is a local base at the point $y = f(x)$. If W is an arbitrary neighborhood of y, then $V_m \subseteq W$ for some m and therefore $f(U_m) \subseteq W$. □

In a metric space M, if $\mathcal{P}_n$ denotes the family of all open balls of diameter less than $\frac{1}{n}$, then the sequence $\{\mathcal{P}_n : n = 1,2,...\}$ constitutes a development for M. A function $f : X \to M$ is said to be *cliquish* if f is a $\mathcal{P}_n$–cliquish function for every $n = 1,2,...$

As we have observed in the above lemma, cliquish functions may be continuous at each point of a "big" set. Let us distinguish some types of functions having plenty of points of continuity as well, and establish some connections with the cliquish ones.

We say that a function $f : X \to Y$ is *r.c. – continuous* if the restriction of f to any regularly closed subset of the space X has a point of continuity.

We say that a function $f : X \to Y$ is *PWD* if f is continuous at each point of a dense subset of X.

LEMMA 2. *Any PWD function is r.c. – continuous.*

PROOF. It follows immediately from the fact that any non-empty regularly closed set has a non-empty interior. □

LEMMA 3. *If $f : X \to Y$ is r.c.–continuous, then the function f is $\mathcal{P}$–cliquish for each open cover $\mathcal{P}$ of the space Y.*

PROOF. Let $\mathcal{P}$ be an open cover of the space Y. To prove that the set $\omega(f;\mathcal{P})$ is dense in X, take arbitrary non-empty open set $U \subseteq X$. Then clU is regularly closed and so the restriction of f to clU, $f|clU$, is continuous at some point, say p. Let V be a member of the cover $\mathcal{P}$ containing $f(p)$. There exists a non-empty open set $G \subseteq X$ such that $p \in G$ and $f(G \cap clU) \subseteq V$. Hence $G \cap U$ is a non-empty open set contained in $\omega(f;\mathcal{P})$. In consequence, $U \cap \omega(f;\mathcal{P}) \neq \emptyset$. □

THEOREM 1. *Let $f : X \to Y$ be a function from a Baire space X into a space Y with a development $\{\mathcal{P}_n : n = 1, 2, ...\}$. The following conditions are equivalent:*
1. f is $\mathcal{P}_n$–cliquish for each $n = 1, 2, ...$;
2. f is r.c.–continuous;
3. f is PWD.

PROOF. The implication (3) $\to$ (2) is in Lemma 2; The implication (2) $\to$ (1) is in Lemma 3; The implication (1) $\to$ (3) is in Lemma 1. □

REMARK 1. The theorem is no longer true without assuming that the domain is a Baire space. There are known examples of *r.c – continuous* functions that are not *PWD*.

Following K. Kuratowski [11], we say that a set F is nowhere dense at a point p if there exists an open neighborhood U of p such that $U \cap F$ is nowhere dense. When preimage by a function of an open set is nowhere dense, even only at some point, the function cannot be continuous. Neither of these types of functions can avoid to posses this deficiency. For our next purposes we will need functions that are *tame*, in the sense that preimages of open sets are nowhere dense at no point.

We say that a function $f : X \to Y$ is *quasi-continuous* if it is *tame* and $\mathcal{P}$*-cliquish* for each open cover $\mathcal{P}$ of Y. To locate quasi-continuous functions among those we have already considered, observe that *r.c.-continuous tame* functions are *quasi-continuous* which, in turn, are $\mathcal{P}$*-cliquish* for each open cover $\mathcal{P}$. Let us consider the following example, due to J. Hoffman-Jorgensen.

EXAMPLE 1. Let X be the square $[-1,1] \times [-1,1]$, let Y to be the product of spaces $Y_{(s,t)}$, where $Y_{(s,t)} = [-1,1]$ for each $(s,t) \in X$, and let $f : X \to Y$ to be the function given by

$$f(u,v)(s,t) = \begin{cases} \dfrac{2(u-s)(v-t)}{(u-s)^2 + (v-t)^2}, \text{ if } (u,v) \neq (s,t), \\ 0, \text{ if } (u,v) = (s,t). \end{cases}$$

One can easily show that f is discontinuous at each point, so f is not r.c.-continuous. However f, treated as a function of two variables, is separately continuous, whence it is quasi-continuous, according to the forthcoming Corollary 1.

We shall employ the following characterization of quasi-continuous functions.

PROPOSITION 1. *Let Y be a regular space. A function $f : X \to Y$ is quasi-continuous if and only if for each open sets $U \subseteq X$ and $V \subseteq Y$, $f(U) \cap V \neq \emptyset$ implies that there exists a non-empty open set $G \subseteq U$ such that $f(G) \subseteq V$.*

PROOF. The implication " $\leftarrow$ " is obvious.

$(\to)$ Let $x \in U$ be such that $f(x) \in V$. There exists an open set W in Y such that $f(x) \in W \subseteq clW \subseteq V$. We set $\mathcal{P} = \{V, Y - clW\}$. Since f is $\mathcal{P}$-*cliquish*, the set $\omega(f;\mathcal{P})$ is dense and open in X. Since f is tame, $U \cap f^{-1}(W)$ cannot be nowhere dense. Hence $\omega(f;\mathcal{P}) \cap U \cap f^{-1}(W)$ is not empty. If z is an element of the latter set, then there exists a neighborhood G of z contained in U and such that $f(G)$ is contained in some member of the cover $\mathcal{P}$. Since $f(G) \cap W \neq \emptyset, f(G) \subseteq Y - clW$ and therefore $f(G) \subseteq V$. □

We are going to use the following three observations pertaining to quasi-continuous functions.

LEMMA 4. *Let $f : X \to Y$ be a quasi-continuous function. If Y is regular and if D is a dense subset of an open set $U \subseteq X$, then $f(U) \subseteq clf(D)$.*

PROOF. Let $x \in U$. To prove that $f(x) \in clf(D)$ take arbitrary open neighborhood V of the point $f(x)$. There exists a non-empty open subset G of U such that $f(G) \subseteq V$. Hence $G \cap D \neq \emptyset$ and therefore $\emptyset \neq f(G \cap D) \subseteq f(G) \cap f(D) \subseteq V \cap f(D)$. □

LEMMA 5. *Let $f : X \to Y$ be a quasi-continuous function into a regular space Y such that $f^{-1}(C)$ is a boundary subset of X whenever C is a boundary subset of Y. Then $f^{-1}(F)$ is a nowhere dense subset of X whenever F is a nowhere dense subset of Y.*

PROOF. Let U be a non-empty open subset of X. If F is a nowhere dense subset of Y, then clF is a boundary subset of Y and therefore U cannot be contained in the set $f^{-1}(clF)$. Hence $f(U)$ must intersect the open set $Y - clF$. There exists a non-empty open set $U_1 \subseteq U$ such that $f(U_1) \subseteq Y - clF$. Hence $U_1 \cap f^{-1}(F) = \emptyset$. □

LEMMA 6. *Let $f : X \to Y$ be a function that is not quasi-continuous. If the space Y is completely regular, then there exists a continuous function $g : Y \to [0, 1]$ such that the composition $g \circ f : X \to [0, 1]$ is not quasi-continuous either.*

PROOF. There are open sets U in X and V in Y such that $f(U) \cap V \neq \emptyset$ and yet if G is a non-empty open subset of U, then $f(G) - V \neq \emptyset$. Let x be a point of U such that $f(x) \in V$. There exists a continuous function $g : Y \to [0, 1]$ such that $g(y) = 0$ for y not in V and $g(f(x)) = 1$. Hence $(g \circ f)(U) \cap (0, 1] \neq \emptyset$ and for every non-empty open subset G of U, $(g \circ f)(G) \nsubseteq (0, 1]$. □

At the end we shall remark on functions that are *hereditarily cliquish* or *hereditarily r.c.-continuous* or *hereditarily PWD*, where "*hereditary*" is meant only with respect to closed sets. So, for example, a function (into a metric space) is going to be called *hereditarily cliquish* if its restriction to arbitrary closed subset of the domain is a cliquish function. Note, however, that in this specific case "hereditary" has its ordinary meaning. *Hereditarily cliquish* functions were considered in [5] as "*functions of first class*". *Hereditarily r.c.-continuous* functions are better known as *barely Baire functions* (cf. [12]). Those three types of functions coincide again if their domain is a space that is *hereditarily Baire.*

3. Automatic Quasi-continuity and Continuity of Actions

I. Namioka [13] has proved that if X is Čech complete and Y is a compact, then for each separately continuous function $f : X \times Y \to M$ into a metric space M there exists a residual subset A of X such that f is jointly continuous at $A \times Y$.

The class of spaces for which Namioka's theorem is true for arbitrary compact space Y has been intensively studied (see [15] for a survey of results). Spaces belonging to that class are called *Namioka spaces.* We will use the fact that Namioka spaces are Baire (cf. [16]).

We say that a space Z is of *pointwise κ-type* if each point of Z belongs to a compact subset C of Z which has a base of neighborhoods of cardinality κ, i.e., there exist open sets U_α, $\alpha < \kappa$, such that for each open neighborhood U of C there exists an $\alpha < \kappa$ with $C \subseteq U_\alpha \subseteq U$. Spaces of *pointwise ω-type* are better known as spaces of *pointwise countable type.*

Standard examples of spaces of pointwise countable type include locally compact Hausdorff spaces and spaces of countable character.

LEMMA 7. *Let Y be a regular space and let C be a compact subset of Y having a base of neighborhoods of cardinality κ. If W is an open set in Y and $q \in W \cap C$, then there exists a compact subset K of Y having also a base of neighborhoods of cardinality κ and such that $q \in K \subseteq W \cap C$.*

PROOF. Let $\{W_n : n \in \omega\}$ be a sequence of open sets in Y such that $W_0 = W$ and $q \in W_{n+1} \subseteq clW_{n+1} \subseteq W_n$ for each $n \in \omega$. Then $K = C \cap \bigcap\{clW_n : n \in \omega\}$

satisfies the required properties. To get a base of neighborhoods of K of size κ, take a base $\mathcal{B}$ of neighborhoods of C of cardinality κ, and consider

$$\mathcal{K} = \{U \cap W_n : U \in \mathcal{B} \text{ and } n \in \omega\}.$$

If V is any open neighborhood of K, then by compactness of C there exists $n \in \omega$ such that $clW_n \cap C \subseteq V$. Thus $clW_n - V$ is disjoint from C. Hence, there exists $U \in \mathcal{B}$ such that $U \cap (clW_n - V) = \emptyset$. So that $U \cap W_n \subseteq V$. □

THEOREM 2. *Let $f : X \times Y \to M$ be a separately continuous function into a metric space M. If X is a Namioka space, Y is a regular space of pointwise κ-type and $b(X) > \kappa$, then f satisfies the following condition:*
(∗) If U, V, W are open sets in X, Y, M, respectively, and $(p, q) \in U \times V$ and $f(p, q) \in W$, then there exist non-empty open sets, G in X and H in Y, such that $G \subseteq U$, $q \in H \subseteq V$, and $f(G \times H) \subseteq W$.

PROOF. Let C be a compact subset of Y that has a base of neighborhoods of cardinality κ and $q \in C$. By Lemma 7 and utilizing continuity of the function $f(p, \cdot)$ we can actually construct such a set C that $C \subseteq V$ and $f(\{p\} \times C) \subseteq W$. There exists an open set W_1 in M such that $f(\{p\} \times C) \subseteq W_1 \subseteq clW_1 \subseteq W$. There exists an open set U_1 in X such that $p \in U_1 \subseteq U$ and $f(U_1 \times \{q\}) \subseteq W_1$.

Let us consider the restriction of the function f to a subspace $X \times C$. Since X is a Namioka space, there exists a residual subset A of X such that the function $f \mid X \times C : X \times C \to M$ is continuous at each point of the set $A \times C$. Let $a \in A \cap U_1$. Thus $f \mid X \times C$ is continuous at (a, q) and $f(a, q) \in W_1$. Hence there exist open sets, U_2 in X and V_1 in Y, such that $a \in U_2 \subseteq U_1$, $q \in V_1 \subseteq V$, and $f(U_2 \times (C \cap V_1)) \subseteq W_1$. Applying Lemma 7 again, we construct a compact set K that has a base of neighborhoods of cardinality κ and such that $q \in K \subseteq C \cap V_1$.

Let $\mathcal{B}$ be a base of neighborhoods of K of cardinality κ. For any $B \in \mathcal{B}$ we set

$$E_B = \{x \in U_2 : f(\{x\} \times B) \subseteq W_1\}.$$

Clearly, the sets E_B, $B \in \mathcal{B}$, cover the set U_2. Since $b(X) > \kappa$, one of them, say E_H, is dense in a non-empty open subset G of U_2. By Lemma 4, $f(G \times H) \subseteq clW_1 \subseteq W$ and the proof of (∗) is finished. □

COROLLARY 1. *Let $f : X \times Y \to Z$ be a separately continuous function into a completely regular space Z. If X is a Namioka space, Y is a regular space of pointwise κ-type and $b(X) > \kappa$, then f is quasi-continuous.*

PROOF. If f were not quasi-continuous, then by Lemma 6, there would exist a function $h : X \times Y \to [0, 1]$ that is separately continuous but not quasi-continuous. By Theorem 2, h would have to satisfy condition (∗) which contradicts the characterization of quasi-continuous functions given in Proposition 1. □

COROLLARY 2. *Let $f : X \times Y \to M$ be a separately continuous function into a metric space M. If X is a Namioka space and Y is a regular space of pointwise countable type, then for each y in Y the set*

$$\{x \in X : f \text{ is jointly continuous at } (x, y)\}$$

is a G_δ dense subset of X. In particular, f is PWD.

PROOF. Fix a y in Y. We shall show that for any $\varepsilon > 0$ the set

$$O_\varepsilon = \{x \in X : \omega(f; (x, y)) < \varepsilon\}$$

is dense in X.

Towards this goal, let U be a non-empty open subset of X and pick any point x in U. By condition $(*)$ of Theorem 2 we get the existence of non-empty open sets G and H such that $G \subseteq U$, $y \in H$, and $f(G \times H)$ is contained in the open ball with the center at $f(x, y)$ and of radius ε. Hence $\emptyset \neq G \subseteq O_\varepsilon$.

Since any Namioka space is Baire, the corollary follows. □

Let $(G; \cdot)$ be a multiplicative algebraic group. We say that the group G *acts on a set* Z if there is a function $A : G \times Z \to Z$, called *action*, that satisfies the following conditions:

(i) $A(1, z) = z$ for each $z \in Z$.

(ii) $A(g \cdot h, z) = A(g, A(h, z))$ for all g, h in G and z in Z.

COROLLARY 3. *Let an abelian group G act on set Z and let A be an action. Suppose G and Z are endowed with topologies such that:*

(j) Z is a metric space and G is a Namioka space;

(jj) G is a topological semi-group (i.e., the group operation is separately continuous and the inverse operation is continuous);

(jjj) Action A is separately continuous.
Then action A is jointly continuous.

PROOF. Since Corollary 2 applies to A, we will be done if we can show that ($) If A is continuous at (g, x) then A is continuous at (h, x) for any $h \in G$.

Take any neighborhood W of $A(h, x)$. Let $k = h \cdot g^{-1}$ (and notice that $h = k \cdot g$). Since $A(h, x) = A(k, A(g, x))$, and since $A(k, \cdot)$ is continuous, there exists an open neighborhood V of $A(g, x)$ in Z such that $A(\{k\} \times V) \subseteq W$. There exist an open neighborhood U of x in Z and an open neighborhood P of g in G such that $A(P \times U) \subseteq V$. Thus $k \cdot P$ is an open neighborhood of h in G such that $A((k \cdot P) \times U) = A(\{k\} \times A(P \times U)) \subseteq A(\{k\} \times V) \subseteq W$. □

4. Automatic Quasi-continuity and Uniqueness of Extensions

We shall prove more results on automatic quasi-continuity of separately continuous functions on the product of two spaces. For the sake of the feasibility of generalizing our next results to separately continuous functions on the product of more than two factors we will formulate them in a more general form.

PROPOSITION 2. *Let $f : X \times Y \to Z$ be a function into a regular space Z such that functions $f(\cdot, y)$ are continuous for each y in Y and functions $f(x, \cdot)$ are quasi-continuous for each x in a dense subset D of X. If $\chi(d, D) < b(Y)$ for each d in D, then f is quasi-continuous.*

PROOF. Let $U \times V$ be an open set in $X \times Y$ containing a point (p, q) and let W be an open set in Z containing $f(p, q)$. There exists an open set W_1 in Z such that $f(p, q) \in W_1 \subseteq clW_1 \subseteq W$. There exists an open set U_1 in x such that $p \in U_1 \subseteq U$ and $f(U_1 \times \{q\}) \subseteq W_1$. Let d be in $U_1 \cap D$. Since $f(d, \cdot)$ is quasi-continuous and $f(d, q) \in W_1$, there exists a non-empty open set V_1 in Y such that $f(\{d\} \times V_1) \subseteq W_1$. Let $\{G_\alpha : \alpha < \kappa\}$ be a family of open subsets of U_1 such that $\{G_\alpha \cap D : \alpha < \kappa\}$ form a base at p in the subspace D and $\kappa < b(Y)$. For $\alpha < \kappa$ we set

$$E_\alpha = \{y \in V_1 : f((G_\alpha \cap D) \times \{y\}) \subseteq W_1\}.$$

Clearly, the sets E_α, $\alpha < \kappa$, cover the set V_1. Hence, one of them, say E_β, is dense in a non-empty open subset V_2 of V_1. By Lemma 4, $f(G_\beta \times V_2) \subseteq clW_1 \subseteq W$. □

PROPOSITION 3. *Let $f : X \times Y \to Z$ be a function into a regular space Z such that $f(x, \cdot)$ and $f(\cdot, y)$ are quasi-continuous functions for all x in X and y in Y. Suppose that for each $p \in X$ there exists its open neighborhood U such that $\pi w(U) < b(Y)$. Then f is quasi-continuous.*

PROOF. Let $U \times V$ be an open set in $X \times Y$ containing a point (p, q) and let W be an open set in Z containing $f(p, q)$. There exists an open set W_1 in Z such that $f(p, q) \in W_1 \subseteq clW_1 \subseteq W$. There exists a non-empty open subset V_1 of V such that $f(\{p\} \times V_1) \subseteq W_1$. Let U_1 be an open neighborhood of p that is contained in U and such that $\pi w(U_1) = \kappa < b(Y)$. If $\{G_\alpha : \alpha < \kappa\}$ is a π-base of U_1, then, for $\alpha < \kappa$, we set

$$E_\alpha = \{y \in V_1 : f(G_\alpha \times \{y\}) \subseteq W_1\}.$$

The sets E_α, $\alpha < \kappa$, cover the set V_1. Indeed, let y be in V_1. Since $f(\cdot, y)$ is quasi-continuous and $f(p) \in W_1$, there exists a non-empty open subset G of U_1 such that $f(G \times \{y\}) \subseteq W_1$. Since the family $\{G_\alpha : \alpha < \kappa\}$ is a π-base of U_1, there is $\beta < \kappa$ such that $G_\beta \subseteq G$. Hence $y \in E_\beta$.

Since $\kappa < b(Y)$, one of the sets E_α, say E_o, is dense in a non-empty open subset V_2 of V_1. By Lemma 4, $f(G \times V_2) \subseteq clW_1 \subseteq W$. □

COROLLARY 4. *Let $f : X_1 \times ... \times X_n \times X_{n+1} \to Z$ be a separately quasi-continuous function into a regular space $\dot{Z}$. If each of the spaces X_κ, $k = 1, 2, .., n$, is a Baire space of countable local π-weight, and X_{n+1} is arbitrary Baire space, then f is quasi-continuous.*

PROOF. Let $X = X_1 \times ... \times X_n$ and $Y = X_{n+1}$. The product $X_1 \times ... \times X_j$ is a Baire space of countable local π-weight for each $1 \leq j \leq n$ (cf. [7]). It follows from Proposition 3 by induction that $f(\cdot, y)$ is a quasi-continuous function for each $y \in Y$. Thus the corollary follows from Proposition 3. □

It is well known that continuous functions into Hausdorff spaces are determined by their values on dense subsets of the domain. It is not the case for quasi-continuous functions; let $f(x) = 0$ if $0 \leq x \leq \frac{1}{2}$ and $f(x) = 1$ if $\frac{1}{2} < x \leq 1$; $g(x) = 0$ if $0 \leq x < \frac{1}{2}$ and $g(x) = 1$ if $\frac{1}{2} \leq x \leq 1$. Then both f and g are quasi-continuous functions on the unit interval $[0, 1]$ which agree everywhere but one non-isolated point $\frac{1}{2}$.

A theorem of Sierpinski [17] asserts that separately continuous real functions of finitely many real variables are determined by their values on dense sets. This theorem has been substantially generalized by W. Comfort [4] (and previously, by C. Goffman and C. Neugebauer [8]). We shall use our results on automatic quasi-continuity of separately quasi-continuous functions to strengthen the result of Comfort.

THEOREM 3. *Let $f, g : X_1 \times ... \times X_n \times X_{n+1} \to Z$ be separately quasi-continuous functions into a completely regular space Z, where each of the spaces X_κ, $k = 1, 2, .., n$, is a Baire space of countable local π-weight, and X_{n+1} is arbitrary Baire space. If f and g agree on a dense set of $X_1 \times ... \times X_n \times X_{n+1}$, then they are identical.*

PROOF. Suppose that $f(p) \neq g(p)$ for some p in $X_1 \times ... \times X_n \times X_{n+1}$. Let $h : Z \to [0, 1]$ be a continuous function such that $h(f(p)) = 0$ and $h(g(p)) = 1$. Then both $h \circ f$ and $h \circ g$ were separately quasi-continuous functions. Suppose that $f = g$ on a dense subset D of $X_1 \times ... \times X_n \times X_{n+1}$. Then $k = h \circ g - h \circ f$ were a separately quasi-continuous function from $X_1 \times ... \times X_n \times X_{n+1}$ into the interval $[-1, 1]$ that vanishes on the set D and takes value 1 at p. By Corollary 4, k is quasi-continuous, so there would exist a non-empty open subset U of $X_1 \times ... \times X_n \times X_{n+1}$ such that $k(U) \subseteq (0, 1]$, which would be impossible. □

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Władysław Kulpa
Institute of Mathematics
University of Silesia
Bankowa 14, 40-007 Katowice
Poland
e-mail: kulpa@ux2.math.us.edu.pl

Andrzej Szymanski
Department of Mathematics
Slippery Rock University
Slippery Rock, PA 16057
U.S.A.
e-mail: andrzej.szymanski@sru.edu
http://www.sru.edu/depts/artsci/math/as/index.htm