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Citation style: Kahlig Peter, Matkowski Janusz. (1999). On a functional equation for Dido's problem. "Annales Mathematicae Silesianae" (Nr 13 (1999), s. 167-180).

ON A FUNCTIONAL EQUATION FOR DIDO'S PROBLEM

PETER KAHLIG AND JANUSZ MATKOWSKI

To the memory of Professor György Targonski

Abstract. By some geometrical considerations we formulate a functional equation which is related to the ancient isoperimetric problem of Dido. The continuous solution of this Dido functional equation depends on an arbitrary function. However, we show that in a class of functions of suitable asymptotic behavior at infinity, the Dido functional equation has a one-parameter family of "principal" solutions. Some applications are given.

1. Introduction

We treat Dido's problem in its most common version, namely, to find the regular polygon with largest area for fixed perimeter. Let us give a comment on the motivation. Dido's problem can be taken as a paradigm of an isoperimetric problem, offering a starting point for a branch of the calculus of variations (cf., e.g., [4], [5]). In the present paper, we attempt to solve Dido's problem by functional equation techniques (without resort to variational methods), supposing only some asymptotic behavior of solutions.

The legend of Dido, Queen of Carthage, has a long tradition. (For historical roots cf., e.g., [9].) We adduce a few citations concerning Dido's problem.

(i) Webster's Dictionary [12]: *Dido's problem is the problem of finding a curve of specified length which encloses the maximum area (the curve being a circle). So called from a tale told of Dido who is said to have bargained for*

Received: December 15, 1998 and, in final form, March 9, 1999.
AMS (1991) subject classification: Primary 39B22, 52B60.

an area a skin could cover and to have then cut the skin to enclose a larger area.

(ii) Edna E. Kramer [5], Ch. 23: *Dido, fleeing from the Phoenician city of Tyre ruled by King Pygmalion, her tyrannical brother, and arriving at the site that was to become Carthage, sought to purchase land from the natives. They asserted that they would sell only as much ground as she could surround with a bull's hide. She accepted the terms and made the most of them by cutting a bull's hide into narrow strips which she pieced together to form a single, very long strip. Then, by sheer intuition, she reasoned that the maximum area could be encompassed by shaping the strip into the circumference of a circle. A rigorous mathematical proof that Dido made the optimum choice was not achieved until the nineteenth century.*

(iii) Christopher Marlowe (1564–1593), *Dido, Queen of Carthage*, Act IV:
She crav'd a hide of ground to build a town.

(iv) Titus Livius (59 B.C.–17), *Ab urbe condita*, XXXIV, Cap. 62, 12:
Advenis, quantum secto bovis tergo amplecti loci potuerint, tantum ad urbem communiendam precario datum.

To the newcomers, as much land as they could enclose by a cut bull's hide [ox-hide, cow-hide], was granted to fortify their town.

Comment by J. Briscoe [3]: *The legend was that the Carthaginians were given by the Numidians the amount of land that could be covered by an ox-hide, and, with Punic cunning, cut the hide into strips.*

(v) Publius Vergilius Maro (70 – 19 B.C.), *Aeneis*, I 365–368:
Devenere locos, ubi nunc ingentia cernes
moenia surgentemque novae Carthaginis arcem,
mercatique solum, facti de nomine Byrsam,
taurino quantum possent circumdare tergo.

They came to places, where now you will see huge
walls and the rising citadel of new Carthage,
and purchased ground, called Byrsa from the name of the fact,
as much as they could surround with a bull's hide.

Comment by R. G. Austin [1]: *The Greek βύρσα (byrsa) means a bull's hide; the Greeks identified with it the Phoenician name for the citadel of Carthage, Bosra; and so the aetiological story arose that Virgil follows in his Aeneid I 368.*

In section 2, by utilizing Lemma 1 (which pertains to a relation for triangles), we formulate the Dido functional equation. In section 3, solutions of the Dido functional equation which satisfy an asymptotic condition are discussed; our main results are contained in Theorems 1 and 2. An application to regular polygons, leading to the isoperimetric inequality, is given in section 4.

2. The Dido functional equation

For some geometrical considerations we first need a lemma concerning an elementary (but non-trivial) relation for triangles (cf. Fig. 1).

LEMMA. *Consider a right-angled triangle with sides of length a, b, c, respectively, where*

$$(0 <) \quad \max(a, b) < c \quad (< \infty).$$

From this, construct an isosceles triangle in the following way: modify the sides of length b and c to an (arbitrary) common value

$$(0 <) \quad b' = c' \quad (< \infty);$$

denote the length of the new base side by a', and by m the length of the height of the isosceles triangle, orthogonal to the base side of length a'. Then

$$m = \frac{b+c}{2} \quad \textit{iff} \quad a' = a.$$

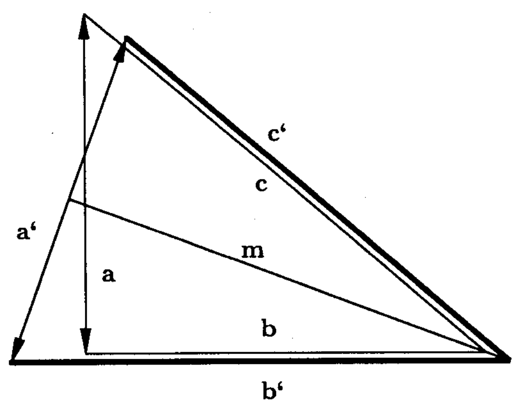

Fig. 1. A relation for triangles

PROOF. Denote by ω the measure of the angle between the sides of length b and c of the right-angled triangle (cf. Fig. 1). Denote by S the area of the corresponding isosceles triangle. Then, obviously,

$$a'm = 2S = [m^2 + (a'/2)^2]\sin(\omega).$$

It follows that

$$\sin(\omega) = a'm/[m^2 + (a'/2)^2].$$

On the other hand we have $\sin(\omega) = a/c$ (from the right-angled triangle) and, consequently,

$$a'm/[m^2 + (a'/2)^2] = a/c.$$

We hence obtain the general result (for arbitrary base side a')

$$m = (a'/a)[\pm\sqrt{c^2 - a^2} + c]/2 = (a'/a)(\pm b + c)/2.$$

Assuming $a' = a$ we get that either $m = (b + c)/2$ or $m = (c - b)/2$. Since $\omega < \pi/2$ we have

$$\tan(\omega/2) = (a/2)/m < 1,$$

and, consequently, $a + b < c$ if $m = (c - b)/2$, which proves that the second case cannot occur.

To prove the converse implication note that

$$4am^2 - 4ca'm + a(a')^2 = 0,$$

and, consequently, either $m = a'(c + b)/(2a)$ or $m = a'(c - b)/(2a)$. Setting $m = (b + c)/2$ implies $a' = a$. Setting $m = (b + c)/2$ in the second relation gives

$$a(b + c) = a'(c - b).$$

Since $(c - b) < a$, this equality implies that $a(b + c) < a'a$ and, consequently,

$$a' > b + c.$$

On the other hand we have

$$\tan(\omega/2) = (a'/2)/m > [(b + c)/2]/[(b + c)/2] = 1,$$

which implies that $\omega > \pi/2$. This contradiction completes the proof.

REMARK 1. Applying the tangent bisection formula, one can give an essentially shorter proof of Lemma 1, as we have

$$(a')/(2m) = (a'/2)/m = \tan(\omega/2) = \sin(\omega)/[1 + \cos(\omega)] = a/(c + b).$$

However, since the tangent bisection (duplication) formula will be obtained later as an equivalent formulation of the functional equation under consideration, this argument is not expedient here. Therefore, the authors preferred a purely geometrical proof of Lemma 1, without resorting to any bisection properties of the trigonometric functions.

REMARK 2. (a) Consider a regular polygon of order $n \in \mathbb{N}_2 := \{2, 3, 4, ...\}$ with fixed perimeter $P \in (0, \infty)$. The length of any segment is $s_n = P/n$.

(b) Consider a regular polygon of order $2n$ with the same fixed perimeter P. The length of any segment is $s_{2n} = P/(2n) = \frac{1}{2}s_n$.

(c) The following relationships hold between the radius r_n of the inner circle and the radius R_n of the outer circle (cf. Fig. 2):

$$R_n = \sqrt{r_n^2 + (s_n/2)^2} = \sqrt{r_n^2 + s_{2n}^2},$$

$$r_{2n} = \frac{r_n + R_n}{2} \qquad \text{(by Lemma 1).}$$

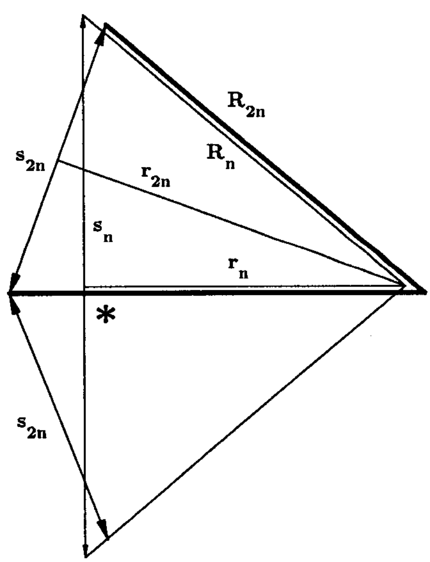

Fig. 2. Elements of regular polygons of order n and $2n$ with fixed perimeter: segments s_n and s_{2n}, radii r_n and r_{2n} of inner circles, radii R_n and R_{2n} of outer circles. (Dido could put a kink in the segment s_n, at $*$, to obtain the two segments s_{2n}, thereby enlarging the area)

Using Remark 2, we can formulate the following

REMARK 3. The radius r of the inner circle of regular polygons with fixed perimeter P obeys the recurrence

$$(1)\qquad r_{2n} = \frac{1}{2}\left[r_n + \sqrt{r_n^2 + (P/(2n))^2}\right], \quad r_n, r_{2n}, P > 0; \quad n \in \mathbb{N}_2.$$

REMARK 4. Let the scaled diameter $2r_n/P$ of a regular polygon of order n (with fixed perimeter P and diameter $2r_n$ of the inner circle) be abbreviated as $f(n) := 2r_n/P$, $f : \mathbb{N}_2 \to [0, \infty)$ denoting a sequence of non-negative values. Then, multiplying (1) by the positive constant $2/P$, we obtain the recurrence ("Dido recurrence")

$$(2)\qquad f(2n) = \frac{1}{2}\left[f(n) + \sqrt{f(n)^2 + n^{-2}}\right], \quad n \in \mathbb{N}_2.$$

Let us attempt to extend the domain of (2) to all $x \geq alpha$ by embedding, replacing f by a corresponding function $\rho : [\alpha, \infty) \to [0, \infty)$, obeying

$$(3)\qquad \rho(2x) = \frac{1}{2}\left[\rho(x) + \sqrt{\rho(x)^2 + x^{-2}}\right], \quad x \geq \alpha$$

(with some $\alpha > 0$, fixed). For (3), an iterative functional equation, we suggest the name "Dido functional equation".

REMARK 5. Note that there is an analogy between the above extension which leads to the Dido functional equation, and the well-known extension of the recurrence $g(n+1) = ng(n)$ to the functional equation $g(x+1) = xg(x)$, $x > 0$, which is used in characterizing the Euler Gamma function.

3. Solutions of the Dido functional equation

To construct the general solution of the Dido functional equation, take an arbitrary function $\rho_0 : [\alpha, 2\alpha] \to [0, \infty)$ such that

$$\rho_0(2\alpha) = \frac{1}{2}\left[\rho_0(\alpha) + \sqrt{\rho_0(\alpha)^2 + \alpha^{-2}}\right],$$

and by induction define a sequence of functions $\rho_n : [2^n\alpha, 2^{n+1}\alpha) \to [0, \infty)$ $(n = 1, 2, ...)$ by the formula

$$\rho_{n+1}(x) := \frac{1}{2}\left[\rho_n(x/2) + \sqrt{\rho_n(x/2)^2 + (x/2)^{-2}}\right],$$
$$x \in [2^{n+1}\alpha, 2^{n+2}\alpha), \quad n = 0, 1, 2, \ldots$$

Then the function $\rho : [\alpha, \infty) \to [0, \infty)$ defined by

$$\rho(x) := \rho_0(x) \;\text{ for }\; x \in [\alpha, 2\alpha),$$

$$\rho(x) := \rho_n(x) \;\text{ for }\; x \in [2^n\alpha, 2^{n+1}\alpha), \quad n = 1, 2, \ldots,$$

satisfies the Dido functional equation. Every solution can be obtained by choosing a suitable function ρ_0. Moreover, it can be easily observed that, if ρ_0 is continuous, then the solution ρ is also continuous.

Thus the continuous solution of the Dido functional equation *depends on an arbitrary function* (cf. Kuczma [6], Kuczma–Choczewski–Ger [7]). However, we shall show that the Dido functional equation has a one-parameter family of "principal" solutions in a class of functions satisfying a general asymptotic condition at infinity. Namely, we have the following

THEOREM 1. *Let $\alpha > 0$ be fixed. Suppose that $\rho : [\alpha, \infty) \to [0, \infty)$ satisfies the Dido functional equation*

$$\rho(2x) = \frac{1}{2}\left[\rho(x) + \sqrt{\rho(x)^2 + x^{-2}}\right], \quad x \geq \alpha. \tag{4}$$

If, for some $a > 0$ and $q > 0$,

$$\rho(x) = a + o(x^{-q}) \quad \text{as} \quad x \to \infty, \tag{5}$$

then

$$\rho(x) = (1/x)\cot(1/(ax)), \quad x \geq \alpha.$$

This function actually satisfies (4) in the interval $[\alpha, \infty)$ if $a \geq 2/(\alpha\pi)$. Moreover, if $a = 2/(\alpha\pi)$, then $\rho(\alpha) = 0$.

PROOF. Define $\phi : (0, 1/\alpha] \to [0, \infty)$ by the formula

$$\phi(x) := x/\rho(1/x), \quad x \in (0, 1/\alpha]. \tag{6}$$

Then

$$\rho(x) = 1/[x\phi(1/x)], \quad x \geq \alpha, \tag{7}$$

and, in view of (5), we have

$$\begin{aligned} x\phi(1/x) &= 1/[a + o(x^{-q})] \\ &= (1/a) - (1/a)o(x^{-q})/[a + o(x^{-q})] = (1/a) + o(x^{-q}) \end{aligned}$$

as $x \to \infty$. It follows that

$$\phi(1/x) = 1/(ax) + o(x^{-1-q}) \;\text{ as }\; x \to \infty,$$

which, of course, means that

$$\phi(x) = bx + o(x^{1+q}) \text{ as } x \to 0, \quad \text{where } b := 1/a. \tag{8}$$

The non-negativity of ρ implies that (4) is equivalent to the functional equation

$$4[\rho(2x)]^2 - 4\rho(2x)\rho(x) = 1/x^2, \quad x \geq \alpha.$$

Hence, after some simple calculations, we get

$$\phi(1/x) = 2\phi(1/(2x))/[1 - (\phi(1/(2x))^2], \quad x \geq \alpha.$$

Thus, replacing x by $1/x$, and making use of (7), we infer that (4) is equivalent to the functional equation

$$\phi(x) = 2\phi(x/2)/[1 - (\phi(x/2))^2], \quad x \in (0, 1/\alpha]. \tag{9}$$

By (8) there exists a function $\psi : (0, 1/\alpha) \to \mathbb{R}$ such that

$$\phi(x) = bx + x^{1+q}\psi(x), \quad \text{where } \lim_{x\to 0} \psi(x) = 0. \tag{10}$$

Inserting this function into (9) it is easy to show that ψ satisfies the functional equation

$$\psi(x) = H(x, \psi(x/2)), \quad x \in (0, 1/\alpha], \tag{11}$$

with $H : D \to R$ given by

$$\begin{aligned} H(x,y) := -\ &[4^q b^3 x^{2-q} + (2^{q+2} + 2^{q+1} b^2 x^2)y + bx^{q+2}y^2]/ \\ &[4^q(b^2x^2 - 4) + 2^{q+1}bx^{q+2}y + x^{2q+2}y^2], \qquad (x,y) \in D, \end{aligned}$$

where

$$\begin{aligned} D := &\{(x,y) \in R^2 : \\ &x \in (0, 1/\alpha];\ -(2/x)^{1+q} - b(2/x)^q < y < (2/x)^{1+q} - b(2/x)^q\}. \end{aligned}$$

Since H is continuously differentiable in D and

$$\lim_{x\to 0} \frac{\partial H}{\partial y}(0,0) = 2^{-q} < 1,$$

we infer that there exist $r > 0, d > 0$ and $c \in (0,1)$ such that

$$|H(x,y) - H(x,z)| \leq c|y - z|, \qquad x \in [0,r], \quad y, z \in [-d, d]. \tag{12}$$

Suppose that $\psi_1, \psi_2 : (0, 1/\alpha] \to \mathbb{R}$ are two solutions of (11) such that

$$\lim_{x\to 0} \psi_1(x) = 0 = \lim_{x\to 0} \psi_2(x).$$

Then there exists a $\delta \in (0, r)$ such that

$$\psi_1(x), \psi_2(x) \in [-d, d], \quad x \in (0, \delta].$$

Hence, applying the Lipschitz condition (12), we have

$$|\psi_1(x) - \psi_2(x)| = |H(x, \psi_1(x/2)) - H(x, \psi_2(x))| \le c|\psi_1(x/2) - \psi_2(x/2)|$$

for all $x \in (0, \delta]$, and, by induction,

$$|\psi_1(x) - \psi_2(x)| \le c^n |\psi_1(x/2^n) - \psi_2(x/2^n)|, \quad n \in \mathbb{N}, \quad x \in (0, \delta].$$

Letting here $n \to \infty$, we obtain

$$\psi_1(x) = \psi_2(x), \quad x \in (0, \delta],$$

and, by (11),

$$\psi_1(x) = \psi_2(x), \quad x \in (0, 1/\alpha].$$

This proves that (11) has at most one solution ψ such that $\lim_{x\to 0} \psi(x) = 0$. It follows that (9) has at most one solution ϕ of the form (10), i.e. such that ϕ satisfies condition (8). Consequently we have shown that the Dido equation (4) has at most one solution ρ satisfying condition (5). Since, evidently, the function

$$\phi(x) = \tan(bx), \quad x \in (0, 1/\alpha]$$

satisfies the functional equation (9) in a right neighborhood of 0, and since $\tan(bx) = bx + o(x^q)$ as $x \to 0$, for every $q \in (0, 2)$, we infer that, for every $b > 0$, the function $\phi(x) = \tan(bx)$ is the only solution of (9) satisfying condition (8). Consequently, (9) has a one-parameter family of solutions. [Of course, the domain of the solution $\phi(x) = \tan(bx)$ depends on b.] Applying formula (6), we get

$$\rho(x) = 1/[x \tan(b/x)] = (1/x)\cot(b/x) = (1/x)\cot(1/(ax)),$$

for all $x \ge \alpha = \alpha(a)$. Since the remaining statements of the theorem are easy to verify, the proof is complete.

REMARK 6. The Dido equation belongs to the broad class of iterative functional equations; in solving them, the iteration theory as developed by

Professor Gy. Targonski (cf. [11]) plays an important role. In the proof of the uniqueness of the solution of (11), we have applied a method developed by one of the authors (J.M., cf. [8] where more general functional equations, and a system of functional equations, are considered).

REMARK 7. Note that Theorem 1 remains true if condition (5) is replaced by the following formally weaker one: there is a $p > 0$ such that

$$\rho(x) = a + O(x^{-p}) \quad \text{as} \quad x \to \infty.$$

REMARK 8. Note that (9) is reducible to the functional equation

$$\gamma(x) = 2\gamma(x/2)^2 - 1, \tag{13}$$

considered in a suitable domain. To show this, define a function γ by

$$\gamma(x) := [1 + \phi(x)^2]^{-1/2}.$$

Since ϕ and γ are positive, we get

$$\phi(x) = \sqrt{1 - \gamma(x)^2}/\gamma(x).$$

Setting this into (9) we obtain

$$\sqrt{1 - \gamma(x)^2}/\gamma(x) = \sqrt{1 - [2\gamma(x/2)^2 - 1]^2}/[2\gamma(x/2)^2 - 1].$$

Since $u \to \sqrt{1 - u^2}/u$ is one-to-one in $(0, 1)$ and $\gamma(x), 2\gamma(x)^2 - 1 \in (0, 1)$, we hence get $\gamma(x) = 2\gamma(x/2)^2 - 1$, which is the desired functional equation. Assuming additionally periodicity of the function γ and applying a result of Sarkovskii [10] (cf. also Kuczma–Choczewski–Ger [7], p. 399), one can replace the regularity condition (8) by the global continuity of γ.

REMARK 9. In the Dido functional equation (4) the argument x appears explicitly; observe that in the equivalent equation (9) the argument does not show up explicitly, so (9) may be called an autonomous functional equation. It is easy to check that the Dido functional equation is also equivalent to the following autonomous functional equation

$$\chi(x/2) = \chi(x) + \sqrt{\chi(x)^2 + 1}. \tag{14}$$

Obviously, this equation is related to the cotangent bisection formula.

REMARK 10. Note that (9) is related to the tangent duplication formula, implying that the Dido functional equation (4) is equivalent to it. [The non-autonomous functional equation (4) can be transformed to an equivalent autonomous functional equation, e.g. (9), (13), or (14).]

REMARK 11. The asymptotic version of (4) for large argument x is

$$x \to \infty : \quad \tilde{\rho}(2x) = \tilde{\rho}(x),$$

leading to the "principal" solution [i.e. satisfying condition (5)] $\tilde{\rho}(x) =$ constant. Essentially the same result can be obtained from the general solution (given in Theorem 1) $\rho(x) = (1/x)\cot(1/(ax))$, namely

$$\rho(\infty) := \lim_{x\to\infty} \rho(x) = a = \text{constant}.$$

4. Application of the Dido recurrence: regular polygons and the isoperimetric inequality

Note that we can adopt the method applied in solving the Dido functional equation (3) to solve the Dido recurrence (2). To show this, put $\mathbb{N}_2^{-1} := \{1/n : n \in \mathbb{N}_2\}$. Replacing f in (2) by the function $\phi : \mathbb{N}_2^{-1} \to [0, \infty)$,

$$\phi(1/n) := 1/[nf(n)], \quad n \in \mathbb{N}_2,$$

we get the equation

$$\phi(1/n) = 2\phi(1/n)/[1 - \phi(1/(2n))^2], \quad n \in \mathbb{N}_2.$$

Repeating in an obvious way arguments used in the proof of Theorem 1, we can prove the following

THEOREM 2. *Let* $f : \mathbb{N}_2 \to [0, \infty)$ *satisfy the recurrence* (2). *If, for some* $a > 0$ *and* $q > 0$,

$$f(n) = a + o(n^{-q}) \quad \textit{as} \quad n \to \infty,$$

then

$$f(n) = (1/n)\cot(1/(an)), \quad n \in \mathbb{N}_2.$$

Moreover, f *satisfies* (2) *in* $\mathbb{N}_2$ *if* $a \geq 1/\pi$, *and, if* $a = 1/\pi$ *then* $f(2) = 0$.

As an immediate consequence we obtain

COROLLARY 1 ("Dido sequence", cf. Fig. 3.). *Suppose that* $f : \mathbb{N}_2 \to [0, \infty)$ *satisfies the conditions of Theorem 2. If* $f(2) = 0$ *then*

$$f(n) = (1/n)\cot(\pi/n), \quad n \in \mathbb{N}_2.$$

Moreover,

$$f(n) < 1/\pi, \quad n \in \mathbb{N}_2,$$

and

$$f(\infty) := \lim_{n\to\infty} f(n) = 1/\pi.$$

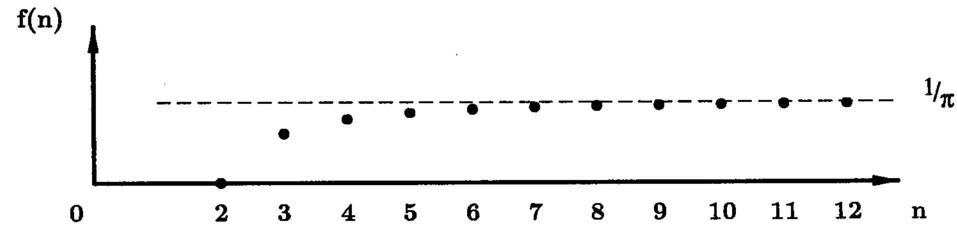

Fig. 3. Dido sequence $f(n) = r_n/(P/2) = A_n/(P/2)^2$, $n = 2, 3, 4, ...$; interpretable as scaled inner radius or scaled area of regular polygons of order n with fixed perimeter P

REMARK 12. According to the geometrical interpretation of the values $f(n)$ in Remark 4, the assumption $f(2) = 0$ means that a regular polygon of order 2 has, by definition, a vanishing inner radius.

By Remark 4 we have $r_n = (P/2)f(n)$. Applying Corollary 1 we get

COROLLARY 2. *The radius of the inner circle of a regular polygon of order* n *with perimeter* P *is*

$$r_n = [P/(2n)]\cot(\pi/n), \quad n = 2, 3, 4, ...$$

The limiting value is

$$r_\infty := \lim_{n\to\infty} [P/(2n)]\cot(\pi/n) = P/(2\pi).$$

Moreover,

$$r_n/P \le 1/(2\pi), \quad n = 2, 3, 4, ...,$$

equality holding for $n \to \infty$ *(circle).*

Since the area of a regular polygon of order n is $A_n = (P/2)r_n = (P^2/4)f(n)$, this corollary implies the following

COROLLARY 3. *The area of a regular polygon of order n is*

$$A_n := [P^2/(4n)]\cot(\pi/n), \quad n = 2, 3, 4, \ldots$$

The limiting value is

$$A_\infty := \lim_{n\to\infty} [P^2/(4n)]\cot(\pi/n) = P^2/(4\pi).$$

Moreover,

$$A_n \leq P^2/(4\pi), \quad n = 2, 3, 4, \ldots,$$

equality holding for $n \to \infty$ (circle).

REMARK 13. The "moreover" part of the above corollary resembles the well-known isoperimetric inequality (cf., e.g., [2], [4]).

REMARK 14. From Corollary 3 there follows $A_2 = 0$. This conforms to the fact that a regular polygon of order 2 has, by definition, a vanishing area. If Dido started with $n = 3$, say (equilateral triangle), she could enlarge the area enclosed (cf. Fig. 3) by generating regular polygons of order $n = 6$ (regular hexagon), $n = 12$ etc., according to the construction of Fig. 2, finally approaching the maximum area for $n \to \infty$ (circle).

Acknowledgments. The authors had the opportunity to discuss the subject of this paper during the ECIT meeting in 1998 at Muszyna-Złockie, Poland. – Philological information was kindly provided by Professor E. Doblhofer. The constructive criticism of a referee is appreciated.

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