Title: Iteration groups of functions satisfying a generalized additivity equation

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ITERATION GROUPS OF FUNCTIONS SATISFYING A GENERALIZED ADDITIVITY EQUATION

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To the memory of Professor György Targonski

Abstract. An elementary proof of the existence of iteration groups consisting of additive bijections of the real line onto itself and containing nonlinear mappings is presented. An embeddability problem of that kind is also considered. These results are used to get a description of some semigroup automorphisms that are embeddable into an iteration group of automorphisms of the same semigroup.

1. Introduction

It is easily seen that a family $\{a^\alpha : \alpha \in \mathbb{R}\}$ of nonzero linear selfmappings of the real line $\mathbb{R}$ forms an iteration group, i.e.

$$a^\alpha \circ a^\beta = a^{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{R},$$

if and only if there exists an exponential function $c : \mathbb{R} \longrightarrow (0, \infty)$ such that

$$a^\alpha(x) = c(\alpha)x \qquad \text{for all} \quad x, \alpha \in \mathbb{R}.$$

In other words, such iteration groups may be identified with nonzero solutions of the functional equation

$$c(\alpha + \beta) = c(\alpha)c(\beta), \quad \alpha, \beta \in \mathbb{R}.$$

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In 1987 Ludwig Reich asked whether one can find an iteration group of additive functions from $\mathbb{R}$ into $\mathbb{R}$ which contains not only linear mappings. This question has been answered in affirmative by M. Sablik [4]. In his approach the structure of orbits of the restriction of an additive bijection transforming a given Hamel basis onto itself played a crucial role. Moreover, only a single example involving some slightly sophisticated considerations has been presented in [4] to show that an iteration group in question can be isomorphic to the additive group $(\mathbb{R}, +)$. In the next section we are going to exhibit a large family of such iteration groups with the aid of entirely elementary means only.

Trying to extend L. Reich's question to iteration groups consisting of solutions of functional equations different from that of additivity (for instance: looking for nontrivial iteration groups of multiplicative functions) we have come to a more general problem of finding suitable groups in the case where the usual addition is replaced by some binary operation $*$ on a given interval $I \subset \mathbb{R}$. More precisely, given a bijection φ of I onto itself, preserving the $*$ operation, i.e. satisfying a generalized Cauchy equation

$$\varphi(x * y) = \varphi(x) * \varphi(y) \qquad \text{for all} \quad x, y \in I,$$

(an automorphism of $(I, *)$), does there exist an iteration group $\{\varphi^\alpha : \alpha \in \mathbb{R}\}$ of automorphisms of $(I, *)$ such that $\varphi^1 = \varphi$? We shall show that the answer to such a question is positive provided $*$ stands for sufficiently regular semigroup operation. A corresponding result as well as numerous concrete examples of iteration groups spoken of are presented in Sections 3 and 4, respectively.

2. Iteration groups of additive bijections

It is well-known (see e. g. M. Kuczma [3, Chapter XII, §5]) that given a Hamel basis H of the real line $\mathbb{R}$ understood as a vector space over the field $\mathbb{Q}$ of all rationals, any bijective selfmapping (permutation) of H has a unique extension to an additive bijection mapping $\mathbb{R}$ onto itself. Since any Hamel basis is necessarily uncountable the collection of all such additive bijections is considerably rich. Moreover, every permutation a_0 of H which satisfies the condition

$$\frac{a_0(h)}{h} \not\equiv \text{const} \quad \text{on} \quad H \tag{1}$$

generates a *nonlinear* (discontinuous) additive bijection $a : H \longrightarrow H$ such that $a|_H = a_0$.

Our first theorem will show that the answer to L. Reich's question is positive. As a matter of fact, much more will be proved: the existence of a pretty rich family of nonlinear additive bijections embeddable into an iteration group of additive bijections which is isomorphic to $(\mathbb{R}, +)$.

THEOREM 1. *Let c be a positive real number that is either transcendental or algebraic of degree greater than 4. Let further H be any Hamel basis of the vector space $\mathbb{R}$ over rationals which contains the set $H_0 := \{c, c^2, c^3, c^4\}$. Given an arbitrary permutation b_1 of the set $H \setminus H_0$ denote by b the additive extension of the bijection $b_0 : H \longrightarrow H$ given by the formula*

$$b_0(h) := \begin{cases} c^2 & \text{for} \quad h = c \\ c^3 & \text{for} \quad h = c^2 \\ c & \text{for} \quad h = c^3 \\ c^4 & \text{for} \quad h = c^4 \\ b_1(h) & \text{for} \quad h \in H \setminus H_0. \end{cases}$$

Then the function $a := b^{-1} \circ cb$ yields a nonlinear additive bijection of $\mathbb{R}$ onto $\mathbb{R}$ that is embeddable into an iteration group of additive bijections $a^\alpha := b^{-1} \circ c^\alpha b, \alpha \in \mathbb{R}$.

PROOF. In the light of the remarks made at the beginning of the present section, the only thing to be proved is the nonlinearity of a. To this end observe that

$$\frac{a(c)}{c} = \frac{b^{-1}(cb(c))}{c} = \frac{b^{-1}(c^3)}{c} = \frac{c^2}{c} = c$$

and

$$\frac{a(c^2)}{c^2} = \frac{b^{-1}(cb(c^2))}{c^2} = \frac{b^{-1}(c^4)}{c^2} = \frac{c^4}{c^2} = c^2 \neq c = \frac{a(c)}{c}.$$

Hence the function $\mathbb{R} \setminus \{0\} \ni t \longmapsto a(t)/t \in \mathbb{R}$ is nonconstant showing that a is nonlinear (discontinuous), as claimed.

Clearly, the function b generating the iteration group in question is not linear either (otherwise $a = a^1$ would be linear).

Now we are going to show that for *every* nonlinear additive bijection $b : \mathbb{R} \longrightarrow \mathbb{R}$ and every positive number $c \neq 1$ the corresponding iteration group $\{a^\alpha : \alpha \in \mathbb{R}\}$ defined by the formula

$$a^\alpha(x) := b^{-1}(c^\alpha b(x)), \quad x, \alpha \in \mathbb{R}, \tag{2}$$

must contain nonlinear members.

THEOREM 2. *Let H and K be two Hamel bases of the vector space $\mathbb{R}$ over the field of rationals and let $b_0 : H \longrightarrow K$ be an arbitrary bijection such that $b_0(h)/h \not\equiv$ const on H. Then the additive extension b of b_0 onto the whole of $\mathbb{R}$ yields a nonlinear additive bijection and for every number $c \in (0,\infty) \setminus \{1\}$ the family $\{a^\alpha : \alpha \in \mathbb{R}\}$ of additive bijections given by* (2) *forms an iteration group containing a nonlinear function.*

PROOF. Actually, the only thing that requires a motivation is the statement that at least one of the functions $a^\alpha = b^{-1}(c^\alpha b(\cdot))$ is nonlinear. To prove it, assume the contrary: for every $\alpha \in \mathbb{R}$ the additive bijection a^α is linear, i.e. there exists a nonzero real constant d_α such that

$$b^{-1}(c^\alpha b(x)) = d_\alpha \cdot x \qquad \text{for all} \quad x \in \mathbb{R}. \tag{3}$$

Put $x_0 := b^{-1}(1)$; then, for every $\alpha \in \mathbb{R}$ one has $b^{-1}(c^\alpha) = x_0 d_\alpha$ or, equivalently, $c^\alpha = b(x_0 d_\alpha)$ whence, by (3),

$$b(x_0 d_\alpha) b(x) = b(d_\alpha x) \qquad \text{for all} \quad x, \alpha \in \mathbb{R}. \tag{4}$$

On the other hand, setting $D := \{x_0 d_\alpha : \alpha \in \mathbb{R}\}$ we infer that $b(D) = \{c^\alpha : \alpha \in \mathbb{R}\} = (0,\infty)$. The injectivity of b jointly with its oddness imply now that $D \cup (-D) \cup \{0\} = \mathbb{R}$ and since equation (4) states that

$$b(d)b(x) = b\left(\frac{1}{x_0} dx\right) \qquad \text{for all} \quad x \in \mathbb{R}, \quad d \in D,$$

we conclude that

$$b(y)b(x) = b(\frac{1}{x_0} yx) \qquad \text{for all} \quad x, y \in \mathbb{R}.$$

This means that an additive bijection $B : \mathbb{R} \longrightarrow \mathbb{R}$ given by the formula $B(x) := b(x_0 x), x \in \mathbb{R}$, is multiplicative, i.e.

$$B(x)B(y) = B(xy) \qquad \text{for all} \quad x, y \in \mathbb{R}.$$

The only selfmappings of $\mathbb{R}$ that are both additive and multiplicative are 0 and identity (see e.g. M. Kuczma [3, Chapter XIV, §4, Theorem 1]. Therefore, B being bijective must be the identity function which implies that $b(x) = (x_0)^{-1}x, x \in \mathbb{R}$, whence, by assumption, const $\not\equiv b_0(h)/h \equiv b(h)/h \equiv (x_0)^{-1}$ on H, a contradiction. Thus the proof has been completed.

3. Embeddable semigroup automorphisms

From the algebraic point of view, additive bijections of $\mathbb{R}$ onto $\mathbb{R}$ constitute automorphisms of the additive group of all real numbers. Replacing the real line by an interval and the usual addition by an abstract semigroup operation we may ask whether a given automorphism of that semigroup is embeddable into an iteration group of semigroup automorphisms. The following theorem gives sufficient conditions for a semigroup in question to get the desired result.

THEOREM 3. *Let $I \subset \mathbb{R}$ be an interval of positive length and let $* : I \times I \longrightarrow I$ be a continuous, associative and cancellative binary operation. Then an automorphism φ of the semigroup $(I, *)$ is embeddable into an iteration group $\{\varphi^\alpha : \alpha \in \mathbb{R}\}$ of automorphisms of $(I, *)$ if and only if either*

(i) *I is arbitrary and $\varphi = id_I$ (the identity function on I),*
or

(ii) *I is half-open and $\varphi(u) = \psi^{-1}(c\psi(u)), u \in I$, where ψ is a strictly increasing bijection of I onto an interval $J \in \{(-\infty, 0], [0, \infty)\}$, and $c \neq 1$ is a positive constant,*
or

(iii) *I is open and $\varphi(u) = \psi^{-1}(d+\psi(u)), u \in I$, or $\varphi(u) = \psi^{-1} \circ a \circ \psi(u), u \in I$, where ψ is a strictly increasing bijection of I onto $\mathbb{R}$, $d \neq 0$ is a real constant and $a : \mathbb{R} \longrightarrow \mathbb{R}$ is an additive bijection that is embeddable into an iteration group $\{a^\alpha : \alpha \in \mathbb{R}\}$ of additive bijections of the real line onto itself.*

The corresponding iteration groups of semigroup automorphisms are given by the formulas

$$\varphi^\alpha := id_I \quad \text{for all} \quad \alpha \in \mathbb{R},$$
$$\varphi^\alpha := \psi^{-1} \circ c^\alpha \cdot \psi \quad \text{for all} \quad \alpha \in \mathbb{R},$$

$$\varphi^\alpha := \psi^{-1} \circ (d\alpha + \psi) \text{ for all } \alpha \in \mathbb{R}, \quad \text{or} \quad \varphi^\alpha := \psi^{-1} \circ a^\alpha \circ \psi \text{ for all } \alpha \in \mathbb{R},$$

respectively.

PROOF. The main result proved by R. Craigen and Zs. Páles in [2] (see also the references therein) states that under the assumptions we have imposed upon the binary operation $*$, there exists an unbounded interval $J \subset \mathbb{R}$ and a strictly increasing bijection $\psi : I \longrightarrow J$ such that

$$u * v = \psi^{-1}\left(\psi(u) + \psi(v)\right), \quad u, v \in I. \tag{5}$$

Fix an automorphism φ of the semigroup $(I,*)$. Then, for every $u,v\in I$ one has

$$\varphi\left(\psi^{-1}(\psi(u)+\psi(v))\right)=\varphi(u*v)=\varphi(u)*\varphi(v)=\psi^{-1}\left(\psi(\varphi(u))+\psi(\varphi(v))\right).$$

Now, for arbitrarily fixed $s,t\in I$, by setting here $u:=\varphi^{-1}(s)$ and $v:=\varphi^{-1}(t)$, we arrive at

$$\psi\circ\varphi\circ\psi^{-1}\left(\psi\circ\varphi^{-1}(s)+\psi\circ\varphi^{-1}(t)\right)=\psi(s)+\psi(t).$$

Plainly, the superposition $a_0:=\psi\circ\varphi\circ\psi^{-1}$ yields a bijective selfmapping of the interval J and since for every two elements $x,y\in J$ the numbers $s:=\varphi\circ\psi^{-1}(x)$ and $t:=\varphi\circ\psi^{-1}(y)$ belong to I, the latter equation implies that

$$a_0(x+y)=a_0(x)+a_0(y),$$

i.e. the additivity of a_0 on the unbounded interval J. It is well known that in such a case a_0 can uniquely be extended to an additive function $a:\mathbb{R}\longrightarrow\mathbb{R}$.

Let us distinguish two cases:

- one of the endpoints of J is finite;
- $J=(-\infty,\infty)=\mathbb{R}$.

In the first case, since $a(J)=a_0(J)=J$, the function a is upper or lower bounded on J forcing a to be linear: $a(x)=c\cdot x$ for all $x\in J$. Clearly, c has to be positive; otherwise a_0 would fail to be a selfmapping. Thus $\psi\circ\varphi\circ\psi^{-1}(x)=cx$ for every $x\in J$, i.e.

$$\varphi(u)=\psi^{-1}(c\psi(u))\qquad\text{for all}\quad u\in I\quad\text{an for some positive } c.$$

In particular, one has

$$I=\varphi(I)=\psi^{-1}(c\psi(I))=\psi^{-1}(cJ),$$

whence $J=\psi(I)=cJ$, i.e.

$$J=cJ.$$

Denoting by ω the finite endpoint of J we deduce that $\omega=c\omega$ whence $c=1$ or $\omega=0$. The first possibility leads to (i) whereas the other implies that $J\in\{(-\infty,0),(-\infty,0],[0,\infty),(0,\infty)\}$. If J happens to be one of the closed half-lines we arrive at (ii). If $J=(0,\infty)$ then $\tilde{\psi}:=\log\psi$ yields a strictly increasing bijection of I onto $\mathbb{R}$ and a simple calculation shows that

$$\varphi(u)=\psi^{-1}(c\psi(u))=\tilde{\psi}^{-1}(d+\tilde{\psi}(u)),\quad u\in I,$$

where $d:=\log c\neq 0$.

Similarly, if $J = (-\infty, 0)$ then setting $\tilde{\psi} := -\log(-\psi)$ we obtain a strictly increasing bijection of I onto $\mathbb{R}$ leading to the same representation of φ with $d := -\log c \neq 0$. Consequently, we obtain the first one of the two possibilities spoken of in (iii).

In all the cases examined till now the embeddability of the automorphism φ considered into the corresponding iteration group of semigroup automorphisms $\{\varphi^\alpha : \alpha \in \mathbb{R}\}$ is self-evident (results immediately from the representations of φ just derived).

Finally, in the second of the two cases distinguished, we have $\varphi(u) = \psi^{-1} \circ a \circ \psi(u), u \in I$, where ψ is a strictly increasing bijection of I onto $\mathbb{R}$. If φ is embeddable into an iteration group of automorphisms $\varphi^\alpha, \alpha \in \mathbb{R}$, then the transformations $a^\alpha := \psi \circ \varphi^\alpha \circ \psi^{-1}, \alpha \in \mathbb{R}$, are all bijective. Moreover, all of them are additive because the equalities

$$\begin{aligned} a^\alpha(x+y) &= \psi \circ \varphi^\alpha \circ \psi^{-1}(x+y) = \psi \circ \varphi^\alpha(\psi^{-1}(x) * \psi^{-1}(y)) \\ &= \psi\left((\varphi^\alpha \circ \psi^{-1})(x) * (\varphi^\alpha \circ \psi^{-1})(y)\right) \\ &= \psi \circ \varphi^\alpha \circ \psi^{-1}(x) + \psi \circ \varphi^\alpha \circ \psi^{-1}(y) \\ &= a^\alpha(x) + a^\alpha(y), \end{aligned}$$

hold true for all x, y and $\alpha \in \mathbb{R}$.

Now, note that for every x, α and $\beta \in \mathbb{R}$ we have

$$\begin{aligned} a^\alpha(a^\beta(x)) &= a^\alpha(\psi \circ \varphi^\beta \circ \psi^{-1}(x)) \\ &= \psi \circ \varphi^\alpha \circ \psi^{-1} \circ \psi \circ \varphi^\beta \circ \psi^{-1}(x) = \psi \circ \varphi^{\alpha+\beta} \circ \psi^{-1}(x) \\ &= a^{\alpha+\beta}(x), \end{aligned}$$

which states that a is embeddable into an iteration group of additive bijections of the real line onto itself.

Conversely, if that is the case, then setting $\varphi^\alpha := \psi^{-1} \circ a^\alpha \circ \psi$ for $\alpha \in \mathbb{R}$ we may easily verify that this family forms an iteration group of semigroup automorphisms that contains the map $\varphi = \psi^{-1} \circ a \circ \psi = \psi^{-1} \circ a^1 \circ \psi$. This completes the proof.

4. Some examples

In what follows, we shall visualize several applications of Theorem 3 in some concrete situations (disregarding the trivial case (i)). Consecutive semigroup operations $*$ occurring below are numbered by their generators ψ in the sense of (5).

$\left(1 - \frac{1}{u}\right)$ $$u * v := \frac{uv}{u + v - uv}, \quad u, v \in I := (0, 1].$$

An automorphism $\varphi : I \longrightarrow I$ of the semigroup $(I, *)$, given by the formula

$$\varphi(u) := \frac{2u}{u+1}, \quad u \in I,$$

is embeddable into the iteration group

$$\varphi^{\alpha}(u) := \frac{2^{\alpha}u}{(2^{\alpha}-1)u+1}, \quad u \in I, \quad \alpha \in \mathbb{R},$$

of automorphisms of $(I, *)$.

$\left(-1 - \frac{1}{u}\right)$ $$u * v := \frac{uv}{u+v+uv}, \quad u, v \in I := [-1, 0).$$

An automorphism $\varphi : I \longrightarrow I$ of the semigroup $(I, *)$, given by the formula

$$\varphi(u) := \frac{u}{u+2}, \quad u \in I,$$

is embeddable into the iteration group

$$\varphi^{\alpha}(u) := \frac{u}{2^{\alpha}(u+1)-u}, \quad u \in I, \quad \alpha \in \mathbb{R},$$

of automorphisms of $(I, *)$.

$\left(\log \frac{1+u}{1-u}\right)$ $$u * v := \frac{u+v}{1+uv}, \quad u, v \in I := [0, 1).$$

An automorphism $\varphi : I \longrightarrow I$ of the semigroup $(I, *)$, given by the formula

$$\varphi(u) := \frac{2u}{1+u^2}, \quad u \in I,$$

is embeddable into the iteration group

$$\varphi^{\alpha}(u) := \frac{(1+u)^{2^{\alpha}} - (1-u)^{2^{\alpha}}}{(1+u)^{2^{\alpha}} + (1-u)^{2^{\alpha}}}, \quad u \in I, \quad \alpha \in \mathbb{R},$$

of automorphisms of $(I, *)$.

Remarkable is the fact that although the associative operation $(*)$ here as well as the formula defining φ are sensible on the whole positive half-line, the embedding procedure described in Theorem 3 cannot be applied because φ extended in that way fails to be injective.

$\left(1-\frac{1}{u^2}\right)$ $$u * v := \frac{uv}{\sqrt{u^2+v^2-u^2v^2}}, \quad u,v \in I := (0,1].$$

An automorphism $\varphi : I \longrightarrow I$ of the semigroup $(I,*)$, given by the formula

$$\varphi(u) := \frac{u}{\sqrt{2-u^2}}, \quad u \in I,$$

is embeddable into the iteration group

$$\varphi^{\alpha}(u) := \frac{u}{\sqrt{2^{\alpha}+(1-2^{\alpha})u^2}}, \quad u \in I, \quad \alpha \in \mathbb{R},$$

of automorphisms of $(I,*)$.

$\left(\log|_{(0,1]}\right)$ $$u * v := uv, \quad u,v \in I := (0,1].$$

An automorphism $\varphi : I \longrightarrow I$ of the semigroup $(I,*)$, given by the formula

$$\varphi(u) := u^2, \quad u \in I,$$

is embeddable into the iteration group

$$\varphi^{\alpha}(u) := u^{2^{\alpha}}, \quad u \in I, \quad \alpha \in \mathbb{R},$$

of automorphisms of $(I,*)$.

$(\log)$ $$u * v := uv, \quad u,v \in I := (0,\infty).$$

An automorphism $\varphi : I \longrightarrow I$ of the semigroup $(I,*)$, given by the formula

$$\varphi(u) := e^{a(\log u)}, \quad u \in I,$$

where $a = b^{-1} \circ cb$ is an arbitrary additive bijection of $\mathbb{R}$ onto itself "constructed" in Theorem 1, is embeddable into the iteration group

$$\varphi^{\alpha}(u) := \exp \circ b^{-1}\left(c^{\alpha} b(\log u)\right) \quad u \in I, \quad \alpha \in \mathbb{R},$$

of automorphisms of $(I, *)$.

$(\log(1+u))$ $$u * v := u + v + uv, \quad u, v \in I := (-1, \infty).$$

An automorphism $\varphi : I \longrightarrow I$ of the semigroup $(I, *)$, given by the formula

$$\varphi(u) := e^{a \log(1+u)} - 1, \quad u \in I,$$

where $a = b^{-1} \circ cb$ with b standing for an arbitrary additive bijection of $\mathbb{R}$ onto itself and $c \neq 1$ being a positive number (see Theorem 2), is embeddable into the iteration group

$$\varphi^{\alpha}(u) := \exp \circ b^{-1}\left(c^{\alpha} b(\log(1+u))\right) - 1, \quad u \in I, \quad \alpha \in \mathbb{R},$$

of automorphisms of $(I, *)$. An appeal to Theorem 2 shows that in the case where b is nonlinear, some of these automorphisms are extremely irregular.

5. Concluding remarks

1. The group $\{a^{\alpha} : \alpha \in \mathbb{R}\}$ constructed in Theorem 1 *always* contains infinitely many linear mappings. Actually, for every $\alpha \in \frac{1}{\log c} \log(\mathbb{Q} \cap (0, \infty))$ the number c^{α} is rational and therefore

$$a^{\alpha}(x) = b^{-1}(c^{\alpha} b(x)) = c^{\alpha} b^{-1}(b(x)) = c^{\alpha} \cdot x, \quad x \in \mathbb{R}.$$

2. In some cases the assumptions imposed upon a semigroup operation $*$ in Theorem 3 may considerably be relaxed. Recently J. Aczél and Gy. Maksa [1] have proved that given an Abelian group $(I, *), I \subset \mathbb{R}$, admitting a continuous subsemigroup $(I_0, *|_{I_0 \times I_0})$, where $I_0 \subset \mathbb{R}$ is an interval of positive length, that generates $(I, *)$, there exists a bijection $\psi : T \longrightarrow \mathbb{R}$ such that $\psi|_{I_0}$ is continuous and the representation

$$u * v = \psi^{-1}\left(\psi(u) + \psi(v)\right),$$

holds true for every $u, v \in I$.
3. Since each iteration group contains the identity mapping as the neutral element, in case (i) of Theorem 3 the corresponding iteration group may be quite arbitrary.

4. J. Smítal has proved in [5] that there exist Hamel bases of the vector space $\mathbb{R}$ over the rationals which are closed under taking powers with integer exponents, i.e. such that jointly with an element h they contain also the set $\{h^n : n \in \mathbb{Z}\}$. Dealing with such type bases one may considerably simplify the construction described in Theorem 1. We omit here some readable rearrangements (simplifications) to be made in that case. However, applying that bases we restrict essentially the collection of semigroups in question.

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