



You have downloaded a document from  
**RE-BUŚ**  
repository of the University of Silesia in Katowice

**Title:** On a factorization of mappings with a prescribed behaviour of the Cauchy difference

**Author:** Roman Ger

**Citation style:** Ger Roman. (1994). On a factorization of mappings with a prescribed behaviour of the Cauchy difference. "Annales Mathematicae Silesianae" (Nr 8 (1994), s. 141-155).



Uznanie autorstwa - Użycie niekomercyjne - Bez utworów zależnych Polska - Licencja ta zezwala na rozpowszechnianie, przedstawianie i wykonywanie utworu jedynie w celach niekomercyjnych oraz pod warunkiem zachowania go w oryginalnej postaci (nie tworzenia utworów zależnych).



UNIwersYTET ŚLĄSKI  
W KATOWICACH



Biblioteka  
Uniwersytetu Śląskiego



Ministerstwo Nauki  
i Szkolnictwa Wyższego

## ON A FACTORIZATION OF MAPPINGS WITH A PRESCRIBED BEHAVIOUR OF THE CAUCHY DIFFERENCE

ROMAN GER

**Abstract.** We deal with functional congruences of the form

$$f(x + y) - f(x) - f(y) \in U + V$$

where  $U$  and  $V$  are given sets subjected to satisfy some "separability" conditions essentially weaker than that occurring in [4] which proved to be pretty useful especially while investigating various types of Hyers-Ulam stability problems. The goal is to factorize  $f$  into a sum of two functions whose Cauchy differences remain in  $U$  and  $V$ , respectively, or, at least to obtain an approximation of  $f$  by such a sum. An application of the newly established result in that spirit is given. Moreover, a stability result for the celebrated cocycle equation is presented and, finally, the behaviour of mappings whose Cauchy differences fall into a given Hamel basis is described.

**1. Introduction.** Stability problems in the sense of Hyers & Ulam may sometimes be reduced to functional congruences of the form

$$(*) \quad f(x + y) - f(x) - f(y) \in U + V.$$

For instance, in a recent paper of R. Ger & P. Šemrl [4] results on representation of mappings satisfying  $(*)$  were used to investigate various aspects of the stability of exponential functions. K. Baron, A. Simon & P. Volkmann [1] have found another application of those results. For convenience, below we quote explicitly the statement of a basic Theorem 2.1 from [4], using the

---

*Received September 16, 1994.*

AMS (1991) subject classification: Primary 39B52. Secondary 39A11.

following notational convention: for an arbitrary set  $U \subset X$  of an Abelian group  $(X, +)$  we put

$$U^+ := U + U \quad \text{and} \quad U^- := U - U,$$

getting, in particular,  $(U^+)^- = U + U - U - U$ .

Let  $(S, +)$  be a cancellative Abelian semigroup and let  $(X, +)$  be a torsion-free divisible Abelian group. Let further  $U, V \subset X$  be such that

$$(**) \quad (U^+)^- \cap (V^+)^- = \{0\}.$$

Then each function  $f : S \rightarrow X$  fulfilling condition (\*) admits a representation of the form  $f = \alpha + \beta$  where  $\alpha, \beta : S \rightarrow X$  satisfy the relationships

$$\alpha(x + y) - \alpha(x) - \alpha(y) \in U, \quad x, y \in S,$$

and

$$\beta(x + y) - \beta(x) - \beta(y) \in V, \quad x, y \in S,$$

The functions  $\alpha$  and  $\beta$  are determined uniquely up to an additive function.

Obviously, the "separation" condition (\*\*) plays here the crucial role. However, as we shall see later on, in some cases it happens to be too strong. The aim of the present paper is to obtain a version of the result just quoted with condition (\*\*) weakened considerably. For that purpose, a generalization of L. Szekélyhidi's result [8] on the stability of the celebrated cocycle equation will be presented, which might be of independent interest.

**2. Stability of the cocycle equation.** We shall be using the following notation: given a nonempty set  $S$  and a normed linear space  $(X, \|\cdot\|)$ , by  $\mathcal{B}(S, X)$  we denote the real linear space of all bounded functions mapping  $S$  into  $X$ , endowed with the norm:  $\|f\|_\infty := \sup\{\|f(s)\| : s \in S\}$ ,  $f \in \mathcal{B}(S, X)$ ; by  $B(x, \rho)$  (resp.  $B_E(x, \rho)$ ) we mean the ball in  $X$  (resp. in a subspace  $E$  of  $X$ ) centered at  $x$  and having radius  $\rho > 0$ , whereas  $\overline{B}(x, \rho)$  will stand for the closure of  $B(x, \rho)$ .

Let us recall that a semigroup  $(S, +)$  is termed *left* (resp. *right*) *amenable* provided that there exists a real linear functional  $M$  on  $\mathcal{B}(S, \mathbb{R})$  such that

$$\inf f(S) \leq M(f) \leq \sup f(S), \quad f \in \mathcal{B}(S, \mathbb{R}),$$

and  $M$  is *left* (resp. *right*) *invariant* in the sense that

$$(1) \quad M({}_a f) = M(f) \quad (\text{resp. } M(f_a) = M(f))$$

for all  $f \in \mathcal{B}(S, \mathbb{R})$  and all  $a \in S$ ; here  $({}_a f)(x) := f(a + x)$  and  $f_a(x) := f(x + a)$ ,  $x, a \in S$ . It is well-known that any commutative semigroup is amenable.

We begin with the following

**THEOREM 1.** *Let  $(S, +)$  be a left (right) amenable semigroup and let  $(X, \|\cdot\|)$  be a real Banach space which is either*

(i) reflexive

or

(ii) has the Hahn-Banach extension property

or

(iii) forms a boundedly complete Banach lattice with a strong unit  $e$ .

Given a number  $\varepsilon \geq 0$  and a mapping  $F : S \times S \rightarrow X$  such that

$$\|F(x + y, z) + F(x, y) - F(x, y + z) - F(y, z)\| \leq \varepsilon, \quad x, y, z \in S,$$

there exists a function  $f : S \times S \rightarrow X$  such that

$$f(x + y, z) + f(x, y) = f(x, y + z) + f(y, z), \quad x, y, z \in S,$$

and

$$\|F(x, y) - f(x, y)\| \leq \varepsilon, \quad x, y \in S,$$

in cases (i) and (ii), whereas

$$\|F(x, y) - f(x, y)\| \leq c_0 \varepsilon, \quad x, y \in S,$$

with  $c_0 := \inf\{c > 0 : \overline{B}(0, 1) \subset c[-e, e]\}$  in case (iii).

**PROOF.** Without loss of generality we may assume that the semigroup  $(S, +)$  is left amenable. An appeal to author's result [3, Theorem 1] gives the existence of a continuous linear operator  $M : \mathcal{B}(S, X) \rightarrow X$  satisfying the first part of (1) for all  $f \in \mathcal{B}(S, X)$ ,  $a \in S$ , such that  $M(c) = c$  for all  $c \in X$  and

$$(2) \quad \|M\| \leq 1$$

in cases (i) and (ii), whereas

$$(3) \quad \|M\| \leq c_0$$

in case (iii). In what follows, to avoid ambiguities, we shall write  $M_z \varphi(x, y, z)$  in the case where the operator  $M$  is assumed to act on a bounded function  $\varphi(x, y, \cdot)$ . Since, for arbitrarily fixed variables  $x, y$  from  $S$  the function

$$F(x + y, \cdot) - F(x, y + \cdot) - F(y, \cdot) : S \rightarrow X$$

is bounded (by  $\|F(x, y)\| + \varepsilon$ ), the formula

$$(4) \quad f(x, y) := M_z(F(x, y + z) + F(y, z) - F(x + y, z)), \quad x, y \in S,$$

correctly defines a map  $f : S \times S \rightarrow X$ . This map turns out to be a solution of the cocycle equation

$$f(x + y, s) + f(x, y) = f(x, y + s) + f(y, s), \quad x, y, s \in S.$$

Indeed, on account of the linearity of the operator  $M$  and its left invariance property one has

$$\begin{aligned} & f(x, y + s) + f(y, s) - f(x + y, s) \\ &= M_z(F(x, y + s + z) + F(y + s, z) - F(x + y + s, z)) \\ & \quad + M_z(F(y, s + z) + F(s, z) - F(y + s, z)) \\ & \quad - M_z(F(x + y, s + z) + F(s, z) - F(x + y + s, z)) \\ &= M_z(F(x, y + s + z) + F(y, s + z) - F(x + y, s + z)) \\ &= M_z(F(x, y + z) + F(y, z) - F(x + y, z)) \\ &= f(x, y) \end{aligned}$$

for all  $x, y, s \in S$ .

To finish the proof, it remains to observe that by means of the equality  $M(c) = c$  valid for all constant functions  $\varphi(x) = c$ ,  $x \in S$ ,  $c \in S$ , we have

$$\begin{aligned} & \|F(x, y) - f(x, y)\| \\ &= \|M_z(F(x, y)) - M_z(F(x, y + z) + F(y, z) - F(x + y, z))\| \\ &= \|M_z(F(x, y) - F(x, y + z) - F(y, z) + F(x + y, z))\| \\ &\leq \|M_z\| \|F(x, y) - F(x, y + z) - F(y, z) + F(x + y, z)\| \\ &\leq \|M_z\| \cdot \varepsilon, \end{aligned}$$

which gives the estimation desired because of (2) and (3).

To get the "right" version, instead of (4) one has to put

$$f(y, z) := M_x(F(x + y, z) + F(x, y) - F(x, y + z)), \quad y, z \in S.$$

This completes the proof. □

**REMARK 1.** The approximating solution of the cocycle equation need not be unique even in the case of scalar valued mappings. However, with the aid of Theorem 2.1 from L. Székelyhidi's paper [8] one can easily show that the

difference between any two scalar solutions of the cocycle equation that approximate the same (scalar) function  $F$  on a right amenable semigroup  $(S, +)$  and fulfilling the inequality

$$|F(x + y, z) + F(x, y) - F(x, y + z) - F(y, z)| \leq \varepsilon,$$

for every  $x, y, z \in S$ , has to be of the form

$$\varphi(x + y) - \varphi(x) - \varphi(y), \quad x, y \in S,$$

where  $\varphi$ , is a *bounded* scalar function on  $S$ .

REMARK 2. In the case where  $(X, \|\cdot\|) = (\mathbf{C}, |\cdot|)$  Theorem 1 has been proved by L. Székelyhidi [8].

**3. Main result.** Armed with the stability result just established we are able to prove the following

THEOREM 2. Let  $(S, +)$  be a cancellative Abelian semigroup and let  $(X, \|\cdot\|)$  be a real Banach space. Assume that  $U$  and  $V$  are nonempty subsets of  $X$  such that the set

$$B := (U^+)^- \cap (V^+)^-$$

is bounded and put  $c := \sup\{\|b\| : b \in B\}$ . Let  $f : S \rightarrow X$  be such that

$$f(x + y) - f(x) - f(y) \in U + V, \quad x, y \in S.$$

If the spaces  $X_U := \text{cl Lin}U$  and  $X_V := \text{cl Lin}V$  satisfy at least one of the conditions (i), (ii) or (iii) spoken of in Theorem 1 (not necessarily the same), then there exist functions  $\alpha, \beta : S \rightarrow X$  satisfying the relations

$$\alpha(x + y) - \alpha(x) - \alpha(y) \in U + \overline{B}_{X_U}(0, \tilde{c}), \quad x, y \in S,$$

$$\beta(x + y) - \beta(x) - \beta(y) \in V + \overline{B}_{X_V}(0, \tilde{c}), \quad x, y \in S,$$

and such that

$$\|f(x) - (\alpha + \beta)(x)\| \leq 2\tilde{c}$$

for all  $x \in S$ ; here  $\tilde{c}$  stands for  $c$  or  $c_0c$  depending on whether condition (i), (ii) or (iii), respectively, is assumed.

PROOF. There exist functions  $\varphi : S \times S \rightarrow U \subset X_U$  and  $\psi : S \times S \rightarrow V \subset X_V$  such that

$$(5) \quad d(x, y) := f(x + y) - f(x) - f(y) = \varphi(x, y) + \psi(x, y), \quad x, y \in S.$$

Obviously,  $d$  is a symmetric solution to the cocycle equation. Moreover,

$$\begin{aligned} (U^+)^- \ni & \varphi(x+y, z) + \varphi(x, y) - \varphi(x, y+z) - \varphi(y, z) \\ & = d(x+y, z) + d(x, y) - d(x, y+z) - d(y, z) \\ & \quad - \psi(x+y, z) - \psi(x, y) + \psi(x, y+z) + \psi(y, z) \\ & = -\psi(x+y, z) - \psi(x, y) + \psi(x, y+z) + \psi(y, z) \in (V^+)^-, \end{aligned}$$

which states that

$$\varphi(x+y, z) + \varphi(x, y) - \varphi(x, y+z) - \varphi(y, z) \in B,$$

or, equivalently, that

$$\|\varphi(x+y, z) + \varphi(x, y) - \varphi(x, y+z) - \varphi(y, z)\| \leq c,$$

for all  $x, y, z \in S$ . Plainly,  $X_U$  is a Banach space and, by assumption, it does have at least one of the properties (i), (ii) or (iii). So, by Theorem 1 there exists a solution  $\Phi : S \times S \rightarrow X_U$  of the cocycle equation such that

$$(6) \quad \|\varphi(x, y) - \Phi(x, y)\| \leq \tilde{c}, \quad x, y \in S.$$

Now, under the assumptions imposed upon the semigroup  $(S, +)$ , with the aid of M. Hosszú's theorem from [5], we infer that there exists a skew-symmetric and biadditive map  $A : S \times S \rightarrow X_U$  and a map  $\alpha_o : S \rightarrow X_U$  such that

$$(7) \quad \Phi(x, y) = A(x, y) + \alpha_o(x+y) - \alpha_o(x) - \alpha_o(y), \quad x, y \in S.$$

However,

$$\begin{aligned} (U^+)^- \supset U - U \ni & \varphi(x, y) - \varphi(y, x) = d(x, y) - \psi(x, y) - d(y, x) + \psi(y, x) \\ & \in V - V \subset (V^+)^-, \end{aligned}$$

i.e.  $\|\varphi(x, y) - \varphi(y, x)\| \leq c$  for all  $x, y \in S$ , whence, by means of (6), we get

$$\begin{aligned} & \|A(x, y) - A(y, x)\| \\ & = \|\Phi(x, y) - \Phi(y, x)\| \\ & \leq \|\Phi(x, y) - \varphi(x, y)\| + \|\varphi(x, y) - \varphi(y, x)\| + \|\varphi(y, x) - \Phi(y, x)\| \\ & \leq \tilde{c} + c + \tilde{c} \end{aligned}$$

for all  $x, y \in S$ . Thus, the biadditive mapping

$$S \times S \ni (x, y) \mapsto A(x, y) - A(y, x) \in X_U$$

remains bounded which is possible if and only if  $A(x, y) = A(y, x)$  for every  $x, y \in S$ . Since  $A$  is skew-symmetric this implies that  $A = 0$ , and, consequently, in view of (7), we obtain the representation

$$\Phi(x, y) = \alpha_o(x + y) - \alpha_o(x) - \alpha_o(y), \quad x, y \in S.$$

Setting

$$r_\varphi(x, y) := \Phi(x, y) - \varphi(x, y), \quad x, y \in S,$$

by virtue of (6) we obtain

$$r_\varphi(x, y) \in \overline{B}_{X_U}(0, \tilde{c}), \quad x, y \in S,$$

whence

$$(8) \quad \alpha_o(x + y) - \alpha_o(x) - \alpha_o(y) = \Phi(x, y) = \varphi(x, y) + r_\varphi(x, y) \in U + \overline{B}_{X_U}(0, \tilde{c})$$

for all  $x, y \in S$ .

Applying literally the same procedure with respect to the function  $\psi : S \times S \rightarrow V \subset X_V$  (see (5)) we deduce the existence of functions  $\beta_o : S \rightarrow X_V$  and  $r_\psi : S \times S \rightarrow \overline{B}_{X_V}(0, \tilde{c})$ , such that

$$(9) \quad \beta_o(x + y) - \beta_o(x) - \beta_o(y) = \psi(x, y) + r_\psi(x, y) \in V + \overline{B}_{X_V}(0, \tilde{c})$$

for all  $x, y \in S$ . From (8), (9) and (5) it follows that for any  $x, y \in S$ , one has

$$\begin{aligned} f(x + y) - f(x) - f(y) &= \alpha_o(x + y) - \alpha_o(x) - \alpha_o(y) - r_\varphi(x, y) \\ &\quad + \beta_o(x + y) - \beta_o(x) - \beta_o(y) - r_\psi(x, y), \end{aligned}$$

i.e.

$$g(x + y) - g(x) - g(y) = -r_\varphi(x, y) - r_\psi(x, y) \in \overline{B}_{X_U}(0, \tilde{c}) + \overline{B}_{X_V}(0, \tilde{c})$$

for all  $x, y \in S$ , where  $g := f - (\alpha_o + \beta_o)$ . In other words, we have

$$\|g(x + y) - g(x) - g(y)\| \leq 2\tilde{c}, \quad x, y \in S,$$

whence, by J. Rätz's [7] generalization of the classical Hyers-Ulam stability result, there exists (a unique) additive mapping  $a : S \rightarrow X$  such that

$$(10) \quad \|g(x) - a(x)\| \leq 2\tilde{c}, \quad x \in S.$$



Obviously, by virtue of the properties (8) and (9), respectively, functions

$$\alpha := \alpha_o + \frac{1}{2}a \quad \text{and} \quad \beta := \beta_o + \frac{1}{2}a$$

fulfil the conditions

$$\alpha(x+y) - \alpha(x) - \alpha(y) \in U + \overline{B}_{X_U}(0, \tilde{c}), \quad x, y \in S,$$

and

$$\beta(x+y) - \beta(x) - \beta(y) \in V + \overline{B}_{X_V}(0, \tilde{c}), \quad x, y \in S.$$

Finally, in view of (10),

$$(11) \quad \|f(x) - (\alpha + \beta)(x)\| \leq 2\tilde{c}$$

for all  $x, y \in S$ , as claimed. Thus the proof has been completed.  $\square$

**REMARK 3.** Once we have  $c = 0$ , which is equivalent to the statement that  $B = \{0\}$ , relation (11) says that  $f = \alpha + \beta$  along with the relationships

$$\alpha(x+y) - \alpha(x) - \alpha(y) \in U$$

and

$$\beta(x+y) - \beta(x) - \beta(y) \in V$$

valid for all  $x, y \in S$ . Therefore, in such a case, the result just proved reduces itself to Theorem 2.1 from [4], quoted explicitly in the Introduction; however, merely in the case where the target group  $(X, +)$  yields a suitable Banach space.

**REMARK 4.** Unfortunately, among the three properties (i), (ii) and (iii) nothing but reflexivity is inherited by closed subspaces of a given Banach space. This forced us to assume that the subspaces  $X_U$  and  $X_V$  occurring in Theorem 2 enjoy at least one of these properties. Dealing with reflexive spaces we obtain more concise statement which reads as follows.

**THEOREM 3.** *Let  $(S, +)$  be a cancellative Abelian semigroup and let  $(X, \|\cdot\|)$  be a real reflexive normed linear space. Assume that  $U$  and  $V$  are nonempty subsets of  $X$  such that the set*

$$B := (U^+)^- \cap (V^+)^-$$

*is bounded and put  $c := \sup\{\|b\| : b \in B\}$ . Let  $X_U := \text{cl Lin } U$  and  $X_V := \text{cl Lin } V$ . If  $f : S \rightarrow X$  is such that*

$$f(x+y) - f(x) - f(y) \in U + V, \quad x, y \in S,$$

then there exist functions  $\alpha, \beta : S \rightarrow X$  satisfying the relations

$$\alpha(x+y) - \alpha(x) - \alpha(y) \in U + \overline{B}_{X_U}(0, c), \quad x, y \in S,$$

$$\beta(x+y) - \beta(x) - \beta(y) \in V + \overline{B}_{X_V}(0, c), \quad x, y \in S,$$

and such that

$$\|f(x) - (\alpha + \beta)(x)\| \leq 2c$$

for all  $x \in S$ .

**4. An application.** The goal of Theorem 2 was to cover the situation where both of the sets  $U$  and  $V$  under consideration are simultaneously unbounded (otherwise, the requirement for the set  $(U^+)^- \cap (V^+)^-$  to be bounded is automatically satisfied). If that is the case, Theorem 2 reduces the problem to a similar one with one of the new summands simply being a ball (in a suitable subspace) and hence bounded. To visualize the utility of such a method we are going to present the following

**EXAMPLE.** Let us consider any cancellative Abelian semigroup  $(S, +)$  and assume a Banach space  $(X, \|\cdot\|)$  to be the real plane  $\mathbb{R}^2$  endowed with the usual Euclidean norm. Fix arbitrarily an  $\varepsilon$  from the interval  $(0, \frac{1}{20})$  and take

$$U := (\mathbf{Z} + (-\varepsilon, \varepsilon)) \times \{0\}, \quad V := \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, x - \varepsilon < y < x + \varepsilon\},$$

where  $\mathbf{Z}$  stands for the set of all integers. Obviously, both  $U$  and  $V$  are unbounded and the intersection

$$\begin{aligned} B &:= (U^+)^- \cap (V^+)^- \\ &= ((\mathbf{Z} + (-4\varepsilon, 4\varepsilon)) \times \{0\}) \cap \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, x - 4\varepsilon < y < x + 4\varepsilon\} \\ &= (-4\varepsilon, 4\varepsilon) \times \{0\} \end{aligned}$$

is greater than  $\{0\}$ . This prevents us from applying the "old" result recalled in the Introduction. However,  $B$  is bounded (we have here  $c := \sup\{\|b\| : b \in B\} = 4\varepsilon$ ) and we can make use of Theorem 3 getting that any function  $f : S \rightarrow \mathbb{R}^2$  whose Cauchy difference  $f(x+y) - f(x) - f(y)$  stays in  $U + V$  for all  $x, y \in S$  admits an approximation of the form

$$\|f(x) - (\alpha + \beta)(x)\| \leq 8\varepsilon,$$

where  $\alpha, \beta : S \rightarrow \mathbb{R}^2$  satisfy the relations

$$(12) \quad \alpha(x+y) - \alpha(x) - \alpha(y) \in U + ([-4\varepsilon, 4\varepsilon] \times \{0\}), \quad x, y \in S,$$

and

$$(13) \quad \beta(x+y) - \beta(x) - \beta(y) \in V + \overline{B}((0,0), 4\varepsilon), \quad x, y \in S.$$

Relation (12) says that the first real component  $\alpha_1$  of the function  $\alpha$  has to satisfy the congruence

$$\alpha_1(x+y) - \alpha_1(x) - \alpha_1(y) \in \mathbf{Z} + (-5\varepsilon, 5\varepsilon), \quad x, y \in S,$$

whereas the other real component  $\alpha_2$  is simply additive. Since  $5\varepsilon \in (0, \frac{1}{4})$  an appeal to Corollary 2.4 from [4] gives the existence of a function  $p : S \rightarrow \mathbb{R}$  such that

$$(14) \quad p(x+y) - p(x) - p(y) \in \mathbf{Z}, \quad x, y \in S,$$

and

$$(15) \quad |\alpha_1(x) - p(x)| \leq 5\varepsilon, \quad x \in S.$$

An easy calculation shows that relation (13) forces the real components  $\beta_1$  and  $\beta_2$  of the function  $\beta$  to satisfy the functional inequality

$$|(\beta_2 - \beta_1)(x+y) - (\beta_2 - \beta_1)(x) - (\beta_2 - \beta_1)(y)| < 5\varepsilon, \quad x, y \in S,$$

whence, by J. Rätz's [7] generalization of the classical Hyers-Ulam stability result, there exists (a unique) additive mapping  $\delta_o : S \rightarrow \mathbb{R}$  such that

$$\|(\beta_2 - \beta_1)(x) - \delta_o(x)\| \leq 5\varepsilon, \quad x \in S.$$

Thus  $\beta_2 = \beta_1 + \delta_o + \varrho$  with  $|\varrho(x)| \leq 5\varepsilon$ , which implies that

$$\beta(x) = d(x) + (0, \delta_o(x) + \varrho(x)), \quad x \in S,$$

where  $d$  is a function from  $S$  into the main diagonal of  $\mathbb{R}^2$  (we have put  $d(x) := (\beta_1(x), \beta_1(x))$ ,  $x \in S$ ). Setting  $\delta := \alpha_2 + \delta_o$  we arrive at

$$(\alpha + \beta)(x) = d(x) + (\alpha_1(x), \delta(x) + \varrho(x)), \quad x \in S.$$

Summarizing, Theorem 3 implies that function  $f$  in question differs by  $8\varepsilon$  in absolute value from a function of the form

$$S \ni x \mapsto d(x) + (\alpha_1(x), \delta(x) + \varrho(x)), \quad x \in S,$$

where  $d$  maps  $S$  into the main diagonal of  $\mathbb{R}^2$ , function  $\alpha_1 : S \rightarrow \mathbb{R}$  satisfies estimation (15) with  $p : S \rightarrow \mathbb{R}$  fulfilling the congruence (14), whereas  $\delta : S \rightarrow \mathbb{R}$  is additive and  $|\varrho(x)| \leq 5\varepsilon$ , for all  $x \in S$ .

**5. Supplementary results.** Even in the (relatively) simplest case where the separation condition

$$(**) \quad (U^+)^- \cap (V^+)^- = \{0\}$$

holds true, the algebraic sum  $U + V$  may happen to be surprisingly large. Obviously, the larger is that sum the more interesting is a result upon a functional congruence of the form

$$(*) \quad f(x + y) - f(x) - f(y) \in U + V.$$

In the present section we shall discuss some special congruences to that effect.

Fix arbitrarily a nonmeasurable Hamel basis  $H$  of the vector space  $\mathbb{R}$  over the field  $\mathbb{Q}$  of all rationals and assume that  $1 \in H$  (see e.g. M. Kuczma [6]). The set  $H + \mathbb{Q}$  is "large" indeed; for, the one-dimensional inner Lebesgue measure  $(\ell_1)_*$  of its complement vanishes:

$$(\ell_1)_*(\mathbb{R} \setminus (H + \mathbb{Q})) = 0.$$

This results immediately from Smítal's lemma (cf. M. Kuczma [6, Chapter III]) because the one-dimensional outer Lebesgue measure of nonmeasurable Hamel basis  $H$  is necessarily positive and the field  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

Nevertheless, we are able to obtain a remarkably precise description of the solutions of the functional congruence

$$(16) \quad f(x + y) - f(x) - f(y) \in H + \mathbb{Q},$$

for functions  $f$  mapping a cancellative Abelian semigroup  $(S, +)$  into  $\mathbb{R}$ . To this aim, we shall first prove the following

**PROPOSITION.** *Under the hypotheses assumed above a function  $f : S \rightarrow \mathbb{R}$  yields a solution to the functional congruence (16) if and only if*

$$f(x) = a(x) + q(x) + h(x), \quad x \in S,$$

where  $a : S \rightarrow \mathbb{R}$  is additive,  $q$  is a function on  $S$  with rational values only and  $h : S \rightarrow \mathbb{R}$  fulfils the condition

$$(17) \quad h(x + y) - h(x) - h(y) \in H,$$

for all  $x, y \in S$ .

PROOF. The "if" part is trivial. Assume that  $f : S \rightarrow \mathbb{R}$  yields a solution to (16). Clearly,

$$(H^+)^- \cap (Q^+)^- = (H + H - H - H) \cap Q = \{0\},$$

because  $H$  is a basis and  $1 \in H$ . By means of Theorem 2.1 from [4] (see the Introduction),  $f$  admits a factorization of the form  $f = r + h$ , where  $r, h : S \rightarrow \mathbb{R}$  are such that

$$(18) \quad r(x + y) - r(x) - r(y) \in Q, \quad x, y \in S,$$

and the congruence (17) is fulfilled.

Let  $Q^c$  denote the space complementary to  $Q$  in the  $Q$ -vector space  $\mathbb{R}$ . Then  $r$  itself factorizes into the sum  $q + a$  where  $q$  is a function on  $S$  with rational values only and  $a$  maps  $S$  into  $Q^c$ . Substituting that representation into (18) immediately shows that  $a$  has to be additive which finishes the proof.  $\square$

It remains to solve the congruence (17). We are able to do that provided the domain of the unknown function forms an Abelian group.

**THEOREM 4.** *Let  $(G, +)$  be an Abelian group and let  $H$  be a Hamel basis of a (real or complex) Banach space  $(X, \|\cdot\|)$ , understood as vector space over the field  $Q$  of all rationals ( $Q$ -vector space). Assume that a function  $h : G \rightarrow X$  satisfies the functional congruence*

$$h(x + y) - h(x) - h(y) \in H, \quad x, y \in G.$$

*Then there exist two different members  $h_0$  and  $h_1$  of the Hamel basis  $H$ , a scalar function  $\lambda : G \rightarrow [-1, 0]$  and an additive mapping  $A : G \rightarrow X$  such that*

$$(19) \quad h(x) = A(x) - h_0 + \lambda(x)(h_1 - h_0), \quad x \in G,$$

and

$$(20) \quad \lambda(x + y) - \lambda(x) - \lambda(y) \in \{0, 1\}, \quad x, y \in G.$$

*In particular,*

$$h(x + y) - h(x) - h(y) \in \{h_0, h_1\}, \quad x, y \in G,$$

*and the function  $h$  differs from an additive mapping by a function whose values are totally contained in the segment  $[-h_0, -h_1] \subset X$ .*

Conversely, for any two members  $h_0$  and  $h_1$  of the Hamel basis  $H$ , any scalar function  $\lambda : G \rightarrow [-1, 0]$  fulfilling the congruence (20) and any additive mapping  $A : G \rightarrow X$ , function  $h : G \rightarrow X$  given by formula (19) yields a solution of the congruence (17).

PROOF. Given a solution  $h : G \rightarrow X$  of the functional congruence (17) put

$$C(x, y) := h(x + y) - h(x) - h(y), \quad x, y \in G.$$

Plainly, the symmetric function  $C : G \times G \rightarrow H$  given by that formula satisfies the cocycle equation

$$(21) \quad C(x + y, z) + C(x, y) = C(x, y + z) + C(y, z), \quad x, y, z \in G.$$

Setting here  $y = z = 0$  we infer that  $C(x, 0) = C(0, 0) =: h_0 \in H$  for all  $x \in G$ . Defining a function  $c : G \rightarrow H$  by the formula  $c(x) := C(x, -x)$ ,  $x \in G$ , and putting  $y = -x$  in (21) we obtain that

$$(22) \quad h_0 + c(x) = C(x, z - x) + C(z, -x), \quad x, z \in G.$$

Take  $z = x + y$  in (22) getting

$$(23) \quad C(x, y) = h_0 + c(x) - C(x + y, -x), \quad x, y \in G.$$

By virtue of the symmetry of  $C$  we have the equality

$$h_0 + c(x) - C(x + y, -x) = h_0 + c(y) - C(x + y, -y), \quad x, y \in G,$$

i.e.

$$c(x) - c(y) = C(x + y, -x) - C(x + y, -y), \quad x, y \in G.$$

Setting here  $x - y$  in place of  $x$  gives

$$c(x - y) - c(y) = C(x, y - x) - C(x, -y), \quad x, y \in G,$$

whence, by means of (22),

$$c(x - y) - c(y) = h_0 + c(x) - C(y, -x) - C(x, -y), \quad x, y \in G,$$

and, consequently, replacing here  $y$  by  $-y$ , and making use of the fact that  $c$  is an even function

$$c(x + y) + C(-x, -y) + C(x, y) = h_0 + c(x) + c(y), \quad x, y \in G.$$

Bearing in mind that all the summands occurring here are members of the Hamel basis we infer that necessarily

$$(24) \quad c(x+y) \in \{h_o, c(x), c(y)\}, \quad x, y \in G.$$

We are going to show that there exists an element  $h_1 \in H$  such that

$$(25) \quad c(x) \in \{h_o, h_1\}, \quad x, y \in G.$$

To this aim, let us first observe that relation (24) and the evenness of  $c$  imply

$$c(x) \in \{h_o, c(x-y), c(y)\}, \quad x, y \in G,$$

as well as

$$c(y) \in \{h_o, c(x-y), c(x)\}, \quad x, y \in G.$$

Thus, for every  $x, y \in G$  one has  $c(x) = c(x-y) = c(y)$ , provided that each two elements of the set  $\{c(x), c(y), h_o\}$  are different. Obviously, this would lead to a contradiction unless

$$c(x) = c(y) \quad \text{or} \quad c(x) = h_o \quad \text{or} \quad c(y) = h_o \quad \text{for all } x, y \in G,$$

but this means nothing else but (25).

Without loss of generality, in the sequel we may assume that  $h_o \neq h_1$ . From (23) and (25) we deduce that

$$C(x, y) \in \{h_o, h_1\} \quad \text{for all } x, y \in G.$$

This means that for a function  $k : G \rightarrow X$  given by the formula

$$k(x) := h(x) + h_o, \quad x \in G,$$

the functional congruence

$$k(x+y) - k(x) - k(y) \in \{0, h_1 - h_o\}, \quad x, y \in G,$$

is valid. Now, an appeal to G. L. Forti's paper [2] guarantees the existence of a scalar function  $\lambda : G \rightarrow [-1, 0]$  and an additive mapping  $A : G \rightarrow X$  such that

$$h(x) + h_o = k(x) = A(x) + \lambda(x)(h_1 - h_o), \quad x \in G,$$

which jointly with the former congruence forces the Cauchy difference of the function  $\lambda$  to satisfy relationship (20). Since the converse part of the theorem is trivial, the proof has been completed.  $\square$

**REMARK 5.** Without any changes in the proof, instead of a Hamel basis one might consider an arbitrary subset  $B$  of the Banach space in question enjoying the following two properties:

- for each elements  $a, b, c, d \in B$  such that  $a+b = c+d$  one has  $a \in \{c, d\}$ ;
- for each elements  $a, b, c, p, q, r \in B$  such that  $a+b+c = p+q+r$  one has  $a \in \{p, q, r\}$ .

## REFERENCES

- [1] K. Baron, A. Simon and P. Volkmann, *On the Cauchy equation modulo a subgroup*, (to appear).
- [2] G. L. Forti, *On an alternative functional equation related to the Cauchy equation*, *Aequationes Math.* **24** (1982), 195-206.
- [3] R. Ger, *The singular case in the stability behaviour of linear mappings*, *Grazer Mathematische Berichte* **316** (1992), 59-70.
- [4] R. Ger and P. Šemrl, *The stability of the exponential equation*, *Proc. Amer. Math. Soc.* (to appear).
- [5] M. Hosszú, *On the functional equation  $F(x + y, z) + F(x, y) = F(x, y + z) + F(y, z)$* , *Period. Math. Hungar.* **1** (1971), 213-216.
- [6] M. Kuczma, *An introduction to the theory of functional equations and inequalities*, Polish Scientific Publishers & Silesian University, Warszawa-Kraków-Katowice, 1985.
- [7] J. Rätz, *On approximately additive mappings*, *General Inequalities 2*, *Internat. Ser. Numer. Math.*, vol. **47**, Birkhäuser Verlag, Basel, 1980, pp. 233-251.
- [8] L. Székelyhidi, *Stability properties of functional equations in several variables*, Technical Report # **91/02**, Department of Mathematics, Lajos Kossuth University (1991), 1-6.

INSTYTUT MATEMATYKI  
UNIwersytet ŚLĄSKI  
BANKOWA 14  
PL-40-007 KATOWICE, POLAND