ZYGFRYD KOMINEK*

THE RECURRENT SEQUENCES OF INEQUALITIES

Abstract. Consider the sequence of inequalities

$$a_{i,k+1} \leqslant \Gamma\Big(\sum_{j=1}^{n} s_{i,j}(a_{1,k}, \ldots, a_{n,k})\Big) + b_{i,k},$$

$i = 1, 2, \ldots, n$, $k = 0, 1, 2, \ldots$, where Γ is a concave and increasing function and the functions $s_{i,j}$, $i, j = 1, \ldots, n$, satisfy some additional conditions ((2.2) and (2.3)). In this note we give an answer to the following question: under what assumptions regarding the given sequences $\{b_{i,k}\}$ their behaviour implies a similar one for the sequences $\{a_{i,k}\}$.

1. Let $a_k, b_k, k = 0, 1, 2, \ldots,$ be non-negative real numbers. We shall consider the sequence of inequalities in the form

(1.1) $$a_k \leqslant \alpha(a_{k-1}) + b_k, \; k = 1, 2, 3, \ldots,$$

where

(1.2) α is an increasing concave function transforming $[0, \infty)$ into itself satisfying the condition $0 < \alpha(t) < t$ for all $t > 0$.

It follows from J. Matkowski's result ([1], Lemma 4.1) that if $\alpha(t) = s \cdot t$, $0 \leqslant s < 1$, then

a) if the sequence $\{b_k\}$ is bounded then $\{a_k\}$ is bounded;

b) if $\lim\limits_{k\to\infty} b_k = 0$ then $\lim\limits_{k\to\infty} a_k = 0$;

c) if $\sum\limits_{k=0}^{\infty} b_k < \infty$ then $\sum\limits_{k=0}^{\infty} a_k < \infty$.

REMARK 1.1. The conditions (1.1) and (1.2), together with $b_k = 0$, $k = 0, 1, 2, \ldots$, do not imply the convergence of the series $\sum\limits_{k=0}^{\infty} a_k$.

In fact, if $a_k = \dfrac{1}{k+1}$, $k = 0, 1, 2, \ldots$, and

$$\alpha(t) = \begin{cases} 1 & \text{for } t \geqslant 1 \\ \dfrac{k}{k+2} & \text{for } t \in \left[\dfrac{1}{k+1}, \dfrac{1}{k}\right), \\ 0 & \text{for } t = 0 \end{cases}$$

then the conditions (1.1) and (1.2) are fulfilled and $\sum\limits_{k=0}^{\infty} a_k$ is not convergent.

Received July 20, 1984.

AMS (MOS) Subject classification (1980). Primary 39C05. Secondary 40A05.

*Instytut Matematyki Uniwersytetu Śląskiego, Katowice, ul. Bankowa 14, Poland.

REMARK 1.2. *Suppose that* (1.1) *and* (1.2) *are fulfilled. Then if* $\lim_{k\to\infty} b_k = 0$ *then* $\lim_{k\to\infty} a_k = 0$.

Proof. Let $a := \limsup a_k$. By (1.1) we have $0 \leqslant a \leqslant \alpha(a)$, which implies, on account of (1.2), that $a = 0$.

It is easy to show, by induction, the following lemma.

LEMMA 1.1. *If the conditions* (1.1) *and* (1.2) *are fulfilled, then for every non-negative integer k the inequalities*

$$a_{k+1} \leqslant \alpha^{k+1}(a_0) + \sum_{p=0}^{k} \alpha^p(b_{k-p}) \tag{1.3}$$

hold. (Here and further α^s *denotes the s-th iterate of* α*).*

Hence, by (1.2) we have

COROLLARY 1.1. *Let the assumptions* (1.1) *and* (1.2) *be fulfilled. If the sequence* $\left\{\sum_{p=0}^{k} \alpha^p(b_{k-p})\right\}$ *is bounded, then* $\{a_k\}$ *is bounded. If* $\lim_{k\to\infty} \alpha^p(b_{k-p}) = 0$, *then* $\lim_{k\to\infty} a_k = 0$.

THEOREM 1.1. *If* (1.1), (1.2) *are fulfilled and the series* $\sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \alpha^p(b_k)$ *is convergent, then the series* $\sum_{k=0}^{\infty} a_k$ *is convergent, too.*

Proof. By Lemma 1.1, it is enough to show that $\sum_{k=0}^{\infty} \alpha^k(a_0)$ and $\sum_{k=0}^{\infty} \sum_{p=0}^{k-1} \alpha^p(b_{k-1-p})$ are convergent. The convergence of the first series follows from the convergence of $\sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \alpha^p(b_k)$. On account of the equality

$$\sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \alpha^p(b_k) = \sum_{p=0}^{\infty} \sum_{k=p}^{\infty} \alpha^p(b_{k-p}) = \sum_{k=0}^{\infty} \sum_{p=0}^{k} \alpha^p(b_{k-p})$$

we get the convergence of the second series. This completes the proof.

THEOREM 1.2. *Let* (1.1), (1.2) *be fulfilled and the series* $\sum_{p=0}^{\infty} \alpha^p(t)$ *is convergent for some* $t > 0$. *If the sequence* $\{b_k\}$ *is bounded, then* $\{a_k\}$ *is bounded, too.*

Proof. Assume that $b_k \leqslant b$, $k = 0, 1, 2, \ldots$. It follows from the monotonicity of α that $\alpha^p(b_{k-p}) \leqslant \alpha^p(b)$, $p = 0, 1, \ldots, k$, and in virtue of the convergence of $\sum_{p=0}^{\infty} \alpha^p(t)$ we have the boundedness of the sequence $\left\{\sum_{p=0}^{k} \alpha^p(b_{k-p})\right\}$. Now, the assertion of our theorem follows from Corollary 1.1.

2. The results of **1.** may be applied in the case of the sequence of systems of inequalities. Let $a_{ik}, b_{ik}, i = 1, \ldots, n, k = 0, 1, 2, \ldots,$ be non-negative real numbers. We assume that

(2.1) there exist an increasing and concave function $\Gamma:[0, \infty) \to [0, \infty)$ and non-negative functions s_{ij}, $i, j = 1, \ldots, n$, of n variables such that

$$a_{i,k+1} \leqslant \Gamma\left(\sum_{j=1}^{n} s_{i,j}(a_{1,k}, \ldots, a_{n,k})\right) + b_{i,k},$$

$k = 0, 1, 2, \ldots, i = 1, \ldots, n;$

(2.2) there exist positive numbers $r_1, \ldots, r_n$ and a concave function $\gamma:[0, \infty) \to \to [0, \infty)$ such that for all $t_i \geqslant 0$, $i = 1, \ldots, n$, the inequalities

$$\sum_{i=1}^{n} s_{ij}(t_1, \ldots, t_n) r_i \leqslant r_j \gamma(t_j), \quad j = 1, \ldots, n,$$

hold;

(2.3) for each $t > 0$, $0 \leqslant \Gamma(\gamma(t)) < t$.

Defining the function

$$\alpha(t) := \sup\{\Gamma(\gamma(s)) \colon 0 \leqslant s \leqslant t\}$$

we observe, that α is increasing and concave.

REMARK 2.1. *The series $\sum_{p=0}^{\infty} (\Gamma(\gamma))^p(t)$ is convergent iff $\sum_{p=0}^{\infty} \alpha^p(t)$ is convergent. The sequence $\{\sum_{p=0}^{k} (\Gamma(\gamma))^p(t_p)\}$ is bounded iff $\{\sum_{p=0}^{k} \alpha^p(t_p)\}$ is bounded.*

We note, in virtue of (2.2), that we may assume that

$$r_1 + \ldots + r_n = 1. \tag{2.4}$$

Let

$$a_k := \sum_{i=1}^{n} r_i a_{i,k},\ b_k := \sum_{i=1}^{n} r_i b_{i,k},\ k = 0, 1, 2, \ldots. \tag{2.5}$$

On account of (2.1) and (2.2) we have

$$\begin{aligned} a_{k+1} &\leqslant \sum_{i=1}^{n} r_i \Gamma\Big(\sum_{j=1}^{n} s_{i,j}(a_{1,k}, \ldots, a_{n,k})\Big) + b_k \leqslant \\ &\leqslant \Gamma\Big(\sum_{j=1}^{n}\sum_{i=1}^{n} s_{i,j}(a_{1,k}, \ldots, a_{n,k}) r_i\Big) + b_k \leqslant \\ &\leqslant \Gamma(\gamma(a_k)) + b_k \leqslant \alpha(a_k) + b_k. \end{aligned}$$

LEMMA 2.1. *If the assumptions* (2.1), (2.2), *and* (2.3) *are fulfilled, then the sequences* $\{a_k\}$ *and* $\{b_k\}$ *defined by* (2.5) *satisfy the condition* (1.1). *Moreover, the function* α *satisfies the condition* (1.2).

Hence, in virtue of Remark 1.2, we get

THEOREM 2.1. *If* (2.1), (2.2), *and* (2.3) *are fulfilled and* $\lim_{k\to\infty} b_{i,k} = 0$, $i = 1, \ldots, n$, *then* $\lim_{k\to\infty} a_{i,k} = 0$, $i = 1, \ldots, n$.

Similarly, according to Lemma 2.1, (2.5), and Theorem 1.2 we get

THEOREM 2.2. *Let* (2.1), (2.2), *and* (2.3) *are fulfilled. If the series* $\sum_{p=0}^{\infty} \alpha^p(t)$ *is convergent for some* $t > 0$, *then the boundedness of the sequences* $\{b_{i,k}\}$, $i = 1, \ldots, n$, implies the boundedness of the sequences $\{a_{i,k}\}$, $i = 1, \ldots, n$.

THEOREM 2.3. *Assume that the assumptions* (2.1), (2.2), *and* (2.3) *are fulfilled. If the series* $\sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \alpha^p(b_{i,k})$, $i = 1, \ldots, n$, *are convergent, then the series* $\sum_{k=0}^{\infty} a_{i,k}$, $i = 1, \ldots, n$ *are convergent.*

Proof. It follows from the monotonicity, concavity of α and (2.4) that

$$\alpha^p\Big(\sum_{i=1}^{n} r_i b_{i,k}\Big) \leqslant \sum_{i=1}^{n} \alpha^p(b_{i,k}).$$

Hence, by assumptions and (2.5) we have the convergence of the series $\sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \alpha^p(b_k)$. According to Lemma 2.1 and Theorem 1.1 we get the assertion of our theorem.

REMARK 2.2. The results contained in Theorems 2.1, 2.2 and 2.3 generalizes some J. Matkowski's result contained in the paper [1] (Lemma 4.1).

In fact, putting $\gamma(t) = s\cdot t$, $0 \leqslant s < 1$, $\Gamma(t) = t$, $s_{i,j}(t_1, \ldots, t_n) = k_{i,j}\cdot t_j$, $k_{i,j} \geqslant 0$, $i,j = 1, \ldots, n$, we observe that the conditions (2.1), (2.2), and (2.3) have the form

$$a_{i,k+1} \leqslant \sum_{j=1}^{n} k_{i,j} a_{j,k} + b_{i,k}, \tag{2.1'}$$

$$\sum_{i=1}^{n} k_{i,j} r_i \leqslant s r_j,\ r_1, \ldots, r_n > 0, \tag{2.2'}$$

whereas the condition (2.2′) is equivalent (see [1, Lemmas 1.1 and 1.2]) to the following one: the matrix $[k_{i,j}]$, $i,j = 1, \ldots, n$, has all the characteristic roots less than one in absolute value.

REFERENCES

[1] J. MATKOWSKI, *Integrable solutions of functional equations*, Dissertationes Math. 77 (1975), 1—68.