ITERATIONS OF MEAN-TYPE MAPPINGS AND INVARIANT MEANS

JANUSZ MATKOWSKI

To the memory of Professor György Targonski

Abstract. It is shown that, under some general conditions, the sequence of iterates of every mean–type mapping on a finite dimensional cube converges to a unique invariant mean–type mapping. Some properties of the invariant means and their applications are presented.

Introduction

The sequence of iterates of a selfmap of a metric space often appears in fixed point theory and, in general, the assumed conditions imply its convergence to a constant map the value of which is a fixed point. In this context the questions whether there are nontrivial selfmaps with non–constant limits of the sequences of iterates, and what are the properties of their limits, seem to be interesting.

To give an answer, in section 2, we consider a class of *mean–type* self–mappings M of a finite dimensional cube I^p, where $I \subseteq \mathbb{R}$ is an interval and $p \geq 2$ a fixed integer, showing that (under some general assumptions) the sequence of iterates $(\mathrm{M}^n)_{n=1}^{\infty}$ converges to a unique non–constant mapping **K** which is an invariant mean–type with respect to M (shortly M–invariant). Since the coordinate functions of M are means, every point of the diagonal of I^p is a fixed point of M. In section 3 we apply these results to determine the limits of the sequence of iterates for some special

Received: March 10, 1999 and, in final form, April 13, 1999.
AMS (1991) subject classification: Primary 26A18, 39B12, 54H25.

classes of mean–type mappings. In section 4 we present some examples of nonexpansive mean–type mappings and we show that the mean–type mapping M= (A, G) (for which the sequence of iterates converges) is neither nonexpansive nor expansive.

The subject considered here is related to the papers by J. Borwein [2], and P. Flor, F. Halter–Koch [4] where a problem concerning some recurrence sequences, posed by J. Aczél [1], was considered.

1. Means and auxiliary results

Let $I \subset \mathbb{R}$ be an interval, and $p \in \mathbb{N}$, $p \geq 2$ fixed. A function $M : I^p \to \mathbb{R}$ is said to be a *mean* on I^p if for all $x = (x_1, \ldots, x_p) \in I^p$,

$$\min(x_1, \ldots, x_p) \leq M(x_1, \ldots, x_p) \leq \max(x_1, \ldots x_p);$$

in particular, $M : I^p \to I$, and, for all $x \in I$,

$$M(x, \ldots, x) = x.$$

A mean M on I^p is called *strict* if whenever $x = (x_1, \ldots, x_p) \in I^p$ such that $x_i \neq x_j$ for some i, $j \in 1, \ldots, p$, then

$$\min(x_1, \ldots, x_p) < M(x_1, \ldots, x_p) < \max(x_1, \ldots x_p);$$

in particular we have the following

REMARK 1. Let $M : I^p \to I$ be a strict mean and let $(x_1, \ldots, x_p) \in I^p$. If

$$M(x_1, \ldots, x_p) = \min(x_1, \ldots, x_p) \quad \textit{or} \quad M(x_1, \ldots, x_p) = \max(x_1, \ldots x_p)$$

then $x_1 = \ldots = x_p$.

LEMMA 1. *Let $p \in \mathbb{N}$, $p \geq 2$, be fixed. Suppose that $M_i : I^p \to \mathbb{R}$, $i = 1, \ldots, p$, are continuous means on I^p such that at most one of them is not strict. Let the functions $M_{i,n} : I^p \to I$, $i = 1, \ldots, p$, $n \in \mathbb{N}$, be defined by*

$$M_{i,1} := M_i, \qquad i = 1, \ldots, p, \tag{1}$$

$$M_{i,n+1}(x_1, \ldots, x_p) := M_i(M_{1,n}(x_1, \ldots, x_p), \ldots, M_{p,n}(x_1, \ldots, x_p)). \tag{2}$$

Then

1^0 *for every* $n \in \mathbb{N}$ *and for each* $i = 1, \ldots, p$, *the function* $M_{i,n}$ *is a continuous mean on* I^p;

2^0 *there is a continuous mean* $K : I^p \to I$ *such that for each* $i = 1, \ldots, p$,

$$\lim_{n \to \infty} M_{i,n}(x_1, \ldots, x_p) = K(x_1, \ldots, x_p), \qquad x_1, \ldots, x_p \in I;$$

3^0 *if* $M_1, \ldots, M_p$ *are strict means, then so is* K.

PROOF. Part 1^0 is obvious. To prove 2^0 assume that, for instance, M_p is strict, and define $\alpha_n, \beta_n : I^p \to I$, $n \in \mathbb{N}$, by

$$\alpha_n := \min(M_{1,n}, \ldots, M_{p,n}), \qquad \beta_n := \max(M_{1,n}, \ldots, M_{p,n}).$$

The functions α_n, β_n are continuous means. Since $M_1, \ldots, M_p$ are means we have

$$\alpha_n \le M_{i,n+1} \le \beta_n, \qquad i = 1, \ldots, p; \ \ n \in \mathbb{N},$$

and, consequently,

$$\alpha_n \le \alpha_{n+1} \le \beta_{n+1} \le \beta_n, \qquad n \in \mathbb{N}.$$

Now we show the following

CLAIM. *For every* $x_1, \ldots, x_p \in I$, *either*
(a) *there is some* $k \in \mathbb{R}$ *such that*

$$\alpha_n(x_1, \ldots, x_p) = \beta_n(x_1, \ldots, x_p), \qquad n \in \mathbb{N}, \ \ n \ge k;$$

or

(b) *for all* $n \in \mathbb{R}$,

$$\alpha_n(x_1, \ldots, x_p) < \alpha_{n+1}(x_1, \ldots, x_p) \ \ or \ \ \beta_{n+1}(x_1, \ldots, x_p) < \beta_n(x_1, \ldots, x_p)$$

This claim is obvious if $x_1 = \ldots = x_p$. Take arbitrary $x_1, \ldots, x_p \in I$ such that $x_i \ne x_j$ for some $i, j \in 1, \ldots, p$. Suppose, for an indirect argument, that the statement (b) does not hold, i.e. that there is a $k \in \mathbb{N}$ such that

$$\alpha_k(x_1, \ldots, x_p) = \alpha_{k+1}(x_1, \ldots, x_p) < \beta_{k+1}(x_1, \ldots, x_p) = \beta_k(x_1, \ldots, x_p).$$

By the definition of α_k and β_k we hence get

$$\begin{aligned} \min(M_{1,k}, \ldots, M_{p,k}) &= \min(M_{1,k+1}, \ldots, M_{p,k+1}) \\ &< \max(M_{1,k+1}, \ldots, M_{p,k+1}) = \max(M_{1,k}, \ldots, M_{p,k}), \end{aligned}$$

and, consequently, there are $i,\ j,\ r,\ s \in \{1,\ldots,p\}$, $i \neq r$, $j \neq s$, such that

$$M_{i,k} = \min\left(M_{1,k},\ldots,M_{p,k}\right) = \min\left(M_{1,k+1},\ldots,M_{p,k+1}\right) = M_{j,k+1}$$
$$< M_{r,k} = \max\left(M_{1,k},\ldots,M_{p,k}\right) = \max\left(M_{1,k+1},\ldots,M_{p,k+1}\right) = M_{s,k+1},$$

(where the values of the occurring functions are taken at the chosen point $(x_1,\ldots,x_p)$). Hence, since

$$M_{j,k+1}\left(x_1,\ldots,x_p\right) = M_j\left(M_{1,k}\left(x_1,\ldots,x_p\right),\ldots,M_{p,k}\left(x_1,\ldots,x_p\right)\right),$$

$$M_{s,k+1}\left(x_1,\ldots,x_p\right) := M_s\left(M_{1,k}\left(x_1,\ldots,x_p\right),\ldots,M_{p,k}\left(x_1,\ldots,x_p\right)\right),$$

and at least one of the means M_j and M_s is strict, applying Remark 1, we infer that

$$M_{1,k}\left(x_1,\ldots,x_p\right) = \ldots = M_{p,k}\left(x_1,\ldots,x_p\right).$$

Hence, by the definition of $M_{i,n+1}$, $i = 1,\ldots,p$, and the fact that the restriction of every mean on I^p to the diagonal of I^p is the identity function on I, we obtain

$$M_{i,n}\left(x_1,\ldots,x_p\right) = M_{j,k}\left(x_1,\ldots,x_p\right), \qquad n \geq k, \quad i,\ j \in \{1,\ldots,p\}.$$

Now the definitions of α_n and β_n give

$$\alpha_n\left(x_1,\ldots,x_p\right) = \beta_n\left(x_1,\ldots,x_p\right), \qquad n \in \mathbb{N},\ n \geq k,$$

showing that relation (a) is true. This completes the proof of our claim.

Since the sequences (α_n) and (β_n) are monotonic and bounded, there exist $\alpha,\ \beta : I^p \to I$ defined by

$$\alpha := \lim_{n\to\infty} \alpha_n, \qquad \beta := \lim_{n\to\infty} \beta_n.$$

We shall show that $\alpha = \beta$. For an indirect argument suppose that there exist $x_1,\ldots,x_p \in I$ such that

$$\alpha\left(x_1,\ldots,x_p\right) < \beta\left(x_1,\ldots,x_p\right).$$

We can assume, without any loss of generality, that, for each $j \in \{2,\ldots,p\}$, M_j is a strict mean. Then for every $j \in \{2,\ldots,p\}$ we have

$$\alpha\left(x_1,\ldots,x_p\right) < M_j\left(\gamma_1\left(x_1,\ldots,x_p\right),\ldots,\gamma_p\left(x_1,\ldots,x_p\right)\right) < \beta\left(x_1,\ldots,x_p\right),$$

where

$$\gamma_i\left(x_1,\ldots,x_p\right) = \alpha\left(x_1,\ldots,x_p\right) \qquad \text{or} \qquad \gamma_i\left(x_1,\ldots,x_p\right) = \beta\left(x_1,\ldots,x_p\right)$$

and $\gamma_r(x_1,\ldots,x_p) \neq \gamma_s(x_1,\ldots,x_p)$ for some r, $s \in \{1,\ldots,p\}$. Take arbitrary positive $\delta > 0$. Then there is $n(\delta)$ such that for all $n \geq n(\delta)$,

$$\alpha(x_1,\ldots,x_p) - \delta < M_{i,n}(x_1,\ldots,x_p) < \beta(x_1,\ldots,x_p) + \delta, \qquad i = 1,\ldots,p,$$

Hence, choosing δ small enough, by the continuity of M_j, we infer that

$$\alpha(x_1,\ldots,x_p) < M_{j,n+1}(x_1,\ldots,x_p) < \beta(x_1,\ldots,x_p),$$

$$j = 2,\ldots p, \quad n \geq n(\delta).$$

It follows that for every $n > n(\delta)$ either

$$\alpha(x_1,\ldots,x_p) < \alpha_n(x_1,\ldots,x_p) < \beta(x_1,\ldots,x_p)$$

or

$$\alpha(x_1,\ldots,x_p) < \beta_n(x_1,\ldots,x_p) < \beta(x_1,\ldots,x_p),$$

which contradicts the definition of α and β. Thus we have shown that

$$\alpha = \beta \qquad \text{in } I^p.$$

Since α_n, β_n are continuous, (α_n) is increasing and (β_n) is decreasing, the function α is lower semicontinuous, and β is upper semicontinuous on I^p. It follows that the function $K : I^p \to I$ defined by

$$K(x_1,\ldots,x_p) := \alpha(x_1,\ldots,x_p), \qquad x_1,\ldots,x_p \in I,$$

is continuous on I^p. It is obvious that K is a mean on I^p. □

LEMMA 2. *Let $p \in \mathbb{N}$, $p \geq 2$, be fixed. Suppose that $M_i : I^p \to \mathbb{R}$, $i = 1,\ldots,p$, are continuous means on I^p such that for some $j \in \{1,\ldots,p\}$, M_j is strict and either*

$$(3) \qquad M_i \leq M_j, \qquad i = 1,\ldots,p,$$

or

$$(4) \qquad M_j \leq M_i, \qquad i = 1,\ldots,p.$$

Then the functions $M_{i,n} : I^p \to I$, $i = 1,\ldots,p$, $n \in \mathbb{N}$, defined by (1)–(2) *in Lemma* 1 *satisfy the conclusions* 1^0–3^0 *of Lemma* 1.

PROOF. Assume that condition (3) is satisfied. Without any loss of generality we can assume that $j = p$, i.e. that

$$M_i \leq M_p, \qquad i = 1,\ldots,p.$$

Part 1^0 is obvious. To prove 2^0 define α_n, β_n, α and β, in the same way as in the proof of Lemma 1. Of course we have

$$\beta_n = M_{p,n}, \qquad \beta_{n+1} \le \beta_n, \qquad (n \in \mathbb{N}), \qquad \beta = \lim_{n\to\infty} M_{p,n}.$$

$$\alpha_n = \min\left(M_{1,n}, \dots, M_{p-1,n}\right), \qquad \alpha_n \le \alpha_{n+1}, \quad (n \in \mathbb{N}),$$

$$\alpha = \lim_{n\to\infty} \alpha_n, \qquad \alpha \le \beta.$$

Suppose that there is a point $(x_1, \dots, x_p) \in I^p$ such that

$$\alpha\left(x_1, \dots, x_p\right) < \beta\left(x_1, \dots, x_p\right).$$

Since M_p is a strict mean we hence get

$$\begin{aligned}\alpha(x_1, \dots, x_p) &< M_p\left(\alpha\left(x_1, \dots, x_p\right), \dots, \alpha\left(x_1, \dots, x_p\right), \beta\left(x_1, \dots, x_p\right)\right)\\ &< \beta\left(x_1, \dots, x_p\right).\end{aligned}$$

Now the continuity of M_p implies that, for sufficiently large n,

$$\alpha\left(x_1, \dots, x_p\right) < M_{p,n}\left(x_1, \dots, x_p\right) < \beta\left(x_1, \dots, x_p\right).$$

This contradiction proves that $\alpha = \beta$. The remaining argument is similar to that of Lemma 1.

Since in the case when condition (4) is satisfied the reasoning is analogous, the proof is completed. □

2. The main results

Let $I \subseteq \mathbb{R}$ be an interval and let $p \in \mathbb{N}$, $p \ge 2$, be fixed. A function M: $I^p \to \mathbb{R}^p$, M$= (M_1, \dots, M_p)$, is called a *mean–type mapping* if each coordinate function M_i, $i = 1, \dots, p$, is a mean on I^p; in particular, M: $I^p \to I^p$. A mean type mapping M$= (M_1, \dots, M_p)$ is *strict* if each of its coordinate functions M_i is a strict mean.

REMARK 2. Note that the restriction of an arbitrary mean–type mapping M: $I^p \to I^p$ to the diagonal of the cube I^p coincides with the identity function i.e., for every $x \in I$,

$$\mathbf{M}\left(x, \dots, x\right) = \left(x, \dots, x\right).$$

It follows that for any function $\mathbf{K}$: $I^p \to I^p$, $\mathbf{K}= (K_1, \ldots, K_p)$, with equal coordinates, i.e. such that $K_1 = \ldots = K_p = K$, we have

$$\mathbf{M} \circ \mathbf{K} = \mathbf{K}.$$

The first result on the convergence of the sequences of iterates of the mean–type mappings reads as follows.

THEOREM 1. *Let an interval $I \subseteq \mathbb{R}$ and $p \in \mathbb{N}$, $p \geq 2$, be fixed. If* $\mathbf{M}$: $I^p \to \mathbb{R}^p$, $M = (M_1, \ldots, M_p)$, *is a continuous mean–type mapping such that at most one of the coordinate means M_i is not strict, then:*

1^0 *for every $n \in \mathbb{N}$, the n–th iterate of $\mathbf{M}$ is a mean–type mapping;*

2^0 *there is a continuous mean $K : I^p \to I$ such that the sequence of iterates $(\mathbf{M}^n)_{n=1}^{\infty}$ converges (pointwise) to a continuous mean–type mapping* $\mathbf{K}$: $I^p \to I^p$, $\mathbf{K}= (K_1, \ldots, K_p)$, *such that*

$$K_1 = \ldots = K_p = K;$$

3^0 $\mathbf{K}$ *is an $\mathbf{M}$–invariant mean–type mapping i.e.,*

$$\mathbf{K} \circ \mathbf{M} = \mathbf{K},$$

or, equivalently, the mean K is $\mathbf{M}$–invariant i.e., for all $x_1, \ldots, x_p \in I$,

$$K\left(M_1(x_1, \ldots, x_p), \ldots, M_p(x_1, \ldots, x_p)\right) = K\left(x_1, \ldots, x_p\right);$$

4^0 *a continuous $\mathbf{M}$–invariant mean–type mapping is unique;*

5^0 *if $\mathbf{M}$ is a strict mean–type mapping then so is $\mathbf{K}$;*

6^0 *if $I = (0, \infty)$ and $\mathbf{M}$ is positively homogeneous, then $\mathbf{K}$ is positively homogeneous.*

PROOF. Define $M_{i,n} : I^p \to I, i = 1, \ldots, p,\ n \in \mathbb{N}$, by formulas (1)-(2). By induction it is easy to verify that

$$\mathbf{M}^n = (M_{1,n}, \ldots, M_{p,n}), \qquad n \in \mathbb{N}.$$

Now, applying Lemma $1.1^0 - 2^0$, we get the conclusions 1^0 and 2^0. Thus, for all $(x_1, \ldots, x_p) \in I^p$, we have

$$\mathbf{K}(x_1, \ldots, x_p) = \lim_{n\to\infty} \mathbf{M}^n(x_1, \ldots, x_p).$$

Hence, making use of (2) and the continuity of $\mathbf{K}$, we get

$$\mathbf{K} = \lim_{n\to\infty} \mathbf{M}^{n+1} = \mathbf{M}(\lim_{n\to\infty} \mathbf{M}^n) = \mathbf{M} \circ \mathbf{K}.$$

Since $\mathbf{K} = (K_1, \ldots, K_p)$ where $K_1 = \ldots = K_p = K$, this relation is equivalent to

$$K(M_1(x_1, \ldots, x_p), \ldots, M_p(x_1, \ldots, x_p)) = K(x_1, \ldots, x_p),$$

for all $(x_1, \ldots, x_p) \in I^p$, and the proof of 3^0 is completed.

To prove 4^0 take an arbitrary continuous mean-type mapping $\mathbf{L}: I^p \to I^p$ that is $\mathbf{M}$ -invariant. Thus we have $\mathbf{L} = \mathbf{L} \circ \mathbf{M}$, and, by an obvious induction,

$$\mathbf{L} = \mathbf{L} \circ \mathbf{M}^n, \qquad n \in \mathbb{N}.$$

Hence, letting $n \to \infty$, making use of 2^0 and the continuity of $\mathbf{L}$ gives

$$\mathbf{L} = \lim_{n\to\infty} \mathbf{L} \circ \mathbf{M}^n = \mathbf{L} \circ (\lim_{n\to\infty} \mathbf{M}^n) = \mathbf{L} \circ \mathbf{K}.$$

Since $\mathbf{K} = (K, \ldots, K)$, in view of Remark 2, we hence get $\mathbf{L}=\mathbf{K}$ which proves the desired uniqueness of the $\mathbf{M}$ -invariant mean.

Part 5^0 is an immediate consequence of Lemma 1.3^0. Since part 6^0 is obvious, the proof is completed.

REMARK 3. The assumption of Theorem 1 that at most one of the means $M_1, .., M_p$ is not strict is essential. To show this consider the following

EXAMPLE 1. Take $p = 3$ and define $L, M, N : \mathbb{R}^3 \to \mathbb{R}$ by

$$L(x, y, z) := \min(x, y, z), \ M(x, y, z) := \frac{x + y + z}{3}, \ N(x, y, z) := \max(x, y, z).$$

Then $\mu := (L + N)/2$ is a mean and for all $x, y, z \in \mathbb{R}$,

$$L(x, y, z) = L(L(x, y, z), \mu(x, y, z), N(x, y, z))$$

$$\mu(x, y, z) = M(L(x, y, z), \mu(x, y, z), N(x, y, z))$$

$$N(x, y, z) = N(L(x, y, z), \mu(x, y, z), N(x, y, z)).$$

Thus, setting $\mathbf{M} := (L, M, N)$ and $\mathbf{K} := (L, \mu, N)$, we have, $\mathbf{K} = \mathbf{M} \circ \mathbf{K}$, i.e. $\mathbf{K}$ is an $\mathbf{M}$-invariant mean-type mapping. However the coordinate means of $\mathbf{K}$ are not equal.

In Theorem 1 we assume that only one of the means $M_1, \ldots, M_p$ is not strict. The next result shows that, under some additional conditions, this assumption can be essentially relaxed.

THEOREM 2. *Let $p \in \mathbb{N}$, $p \geq 2$, be fixed. Suppose that $\mathbf{M}$: $I^p \to I^p$, $M = (M_1, \ldots, M_p)$, is a continuous mean-type mapping. Let $(\mathbf{M}^n)_{n=0}^{\infty}$ be the sequence of iterations of $\mathbf{M}$. If there is an $j \in 1, \ldots, p$ such that M_j is strict and either*

$$M_i \leq M_j, \qquad i = 1, \ldots, p, \tag{5}$$

or

$$M_j \leq M_i, \qquad i = 1, \ldots, p, \tag{6}$$

then

1^0 *for every $n \in \mathbb{N}$, the iterate $\mathbf{M}^n$ is a mean type mapping on I^p;*

2^0 *the sequence $(\mathbf{M}^n)_{n=1}^{\infty}$ converges (pointwise) to a mean type mapping $\mathbf{K}$: $I^p \to I^p$, $\mathbf{K} = (K_1, \ldots, K_p)$, such that*

$$K_1 = \ldots = K_p;$$

3^0 $\mathbf{K}$ *is $\mathbf{M}$-invariant i.e.,*

$$\mathbf{K} \circ \mathbf{M} = \mathbf{K};$$

4^0 *a continuous $\mathbf{M}$-invariant mean-type mapping is unique,*

5^0 *if $\mathbf{M}$ is a strict mean-type mapping then so is $\mathbf{K}$;*

6^0 *if $I = (0, \infty)$ and $\mathbf{M}$ is positively homogeneous, then $\mathbf{K}$ is positively homogeneous.*

PROOF. It is enough to apply Lemma 2 and argue along the same line as in the case of Theorem 1.

REMARK 4. Example 1 shows that the existence of a strict coordinate mean of a mean-type mapping M such that either condition (5) or (6) is satisfied is an essential assumption of Theorem 2.

3. Invariant means and applications of main results

According to Theorem 1 and Theorem 2, the problem to determine the limit of the sequence of iterates of a mean-type mapping M reduces to finding an M-invariant mean-type mapping (or an M-invariant mean). To show that this fact can be helpful in determining the limit of the sequence $(\mathbf{M}^n)$ we begin this section by presenting the following

EXAMPLE 2. Take $I = (0, \infty)$ and $p = 2$. Let M: $I^2 \to I^2$ be defined by M$= (A, H)$, where A and H are respectively the arithmetic and harmonic means:

$$A(x,y) = \frac{x+y}{2}, \qquad H(x,y) = \frac{2xy}{x+y}, \qquad x, y \in \mathrm{I}.$$

By Theorem 1 there exists a unique mean-type mapping K: $I^2 \to I^2$ which is invariant with respect to M. Let G be the geometric mean, $G(x,y) = (xy)^{1/2}, (x, y \in I)$. Since (cf. P. Kahlig, J.Matkowski [5])

$$G(A(x,y), H(x,y)) = \left(\frac{x+y}{2}\frac{2xy}{x+y}\right)^{1/2} = G(x,y), \qquad x, y > 0,$$

G is an M–invariant mean and, by the uniqueness of the invariant mean, we have K$= (G, G)$. Moreover,

$$\lim_{n\to\infty} \mathrm{M}^n(x,y) = \lim_{n\to\infty} \left(\frac{x+y}{2}, \frac{2xy}{x+y}\right)^n = \left(\sqrt{xy}, \sqrt{xy}\right), \qquad x, y > 0.$$

This example can be easily deduced from more general facts presented below as Propositions 1-3 in which we consider some special classes of means.

Given $r \in \mathbb{R}, r \neq 0$, the function $M^{[r]} : (0, \infty)^2 \to (0, \infty)$,

$$M^{[r]}(x,y) := \left(\frac{x^r + y^r}{2}\right)^{1/r}, \qquad x, y > 0,$$

is called the *power mean.*

Now we prove

PROPOSITION 1. *Let $r \in \mathbb{R}, r \neq 0$, be fixed. Then*

$$G(M^{[r]}(x,y), M^{[-r]}(x,y)) = G(x,y), \quad x, y > 0,$$

i.e., for all $r \in \mathbb{R}$, the geometric mean G is invariant with respect to the mean-type mapping M$= (M^{[r]}, M^{[-r]})$. *Moreover,*

$$\lim_{n\to\infty} \mathrm{M}^n = (G, G).$$

PROOF. By simple calculation, we verify the invariance. The remaining part of the proposition follows from Theorem 1. □

For a fixed $r \in \mathbb{R}$ define $D^{[r]} : (0,\infty)^2 \to (0,\infty)$ by

$$D^{[r]}(x,y) := \begin{cases} \frac{x-y}{\log x - \log y}, & r = 0 \\ \frac{r}{r+1}\frac{x^{r+1}-y^{r+1}}{x^r - y^r}, & -1 \neq r \neq 0, \quad (x,y>0). \\ xy\frac{\log x - \log y}{x-y}, & r = -1 \end{cases}$$

$D^{[r]}$ is called the *difference quotient mean*.

PROPOSITION 2. *For all $r \in \mathbb{R}$,*

$$G(D^{[r]}(x,y), D^{[-r-1]}(x,y)) = G(x,y), \qquad x,y > 0,$$

i.e., the geometric mean G is invariant with respect to the mean-type mapping $\mathbf{M} = (D^{[r]}, D^{[-r-1]})$. *Moreover,*

$$\lim_{n\to\infty} \mathbf{M}^n = (G,G).$$

(We omit an easy proof of Proposition 2, as well as Proposition 3, below).

For a fixed $r \in \mathbb{R}$ the function $G^{[r]} : (0,\infty)^2 \to (0,\infty)$ given by

$$G^{[r]}(x,y) := \frac{x^{r+1/2} + y^{r+1/2}}{x^{r-1/2} + y^{r-1/2}}, \qquad x,y > 0,$$

is the *Gini mean* (Bullen–Mitrinović–Vasić [3], p. 189). Note that $G^{[0]} = G$.

PROPOSITION 3. *For all $r \in \mathbb{R}$,*

$$G(G^{[r]}(x,y), G^{[-r]}(x,y)) = G(x,y), \qquad x,y > 0,$$

i.e., the geometric mean G is invariant with respect to the mean-type mapping $\mathbf{M} = (G^{[r]}, G^{[-r]})$. *Moreover,*

$$\lim_{n\to\infty} \mathbf{M}^n = (G,G).$$

In connection with Propositions 1-3 let us note a general

REMARK 5. Let $I \subset (0,\infty)$ be an interval. If $M : I^2 \to I$ is a mean then $N : I^2 \to \mathbb{R}$, defined by

$$N(x,y) := \frac{xy}{M(x,y)}, \qquad x,y \in I,$$

is a mean. Moreover, the geometric mean G is invariant with respect to the mean-type $\mathbf{M}:= (M,N)$, and $\lim_{n\to\infty} \mathbf{M}^n = (G,G)$.

The next result (which is easy to verify) gives a broad class of mean-type mappings $\mathbf{M}: I^2 \to I^2$ for which the $\mathbf{M}$-invariant means are quasi-arithmetic.

PROPOSITION 4. *Let $\phi : I \to \mathbb{R}$ be continuous and strictly monotonic. Suppose that $M : I^2 \to I$, is a mean. Then the function $N : I^2 \to I$ defined by*

$$N(x,y) := \phi^{-1}(\phi(x) + \phi(y) - \phi(M(x,y))$$

is a mean. Moreover, the quasi-arithmetic mean $K : I^2 \to I$, defined by

$$K(x,y) := \phi^{-1}(\frac{\phi(x)+\phi(y)}{2}),$$

is $\mathbf{M}$-invariant for a mean-type mapping $\mathbf{M}= (M,N)$.

Example 2 and Propositions 1-4 were concerned with the case $p = 2$. If $p \geq 3$ the situation is a little more complicated. However, the following counterpart of Proposition 4 is easily verified.

PROPOSITION 5. *Let $p \geq 3, p \in \mathbb{N}$, and a continuous strictly increasing function $\phi : I \to \mathbb{R}$ be fixed. Suppose that $M_i : I^p \to I, i = 1,\ldots,p-1$, are symmetric means which are increasing with respect to each variable. Then the function $M_p : I^p \to I$ defined by*

$$M_p(x_1,\ldots,x_p) := \phi^{-1}(\sum_{i=1}^{p} \phi(x_i) - \sum_{i=1}^{p-1} \phi(M_i(x_1,,\ldots,x_p)))$$

is a mean if, and only if, the following two conditions are satisfied:

(a) *for all $x_2,\ldots,x_p \in I$,*

$$x_2 < \ldots < x_p \Rightarrow \sum_{i=1}^{p-1} \phi(M_i(x_2,x_2,x_3,\ldots,x_p)) \leq \sum_{i=2}^{p} \phi(x_i);$$

(b) *for all $x_1,\ldots,x_{p-1} \in I$,*

$$x_1 < \ldots < x_{p-1} \Rightarrow \sum_{i=1}^{p-1} \phi(x_i) \leq \sum_{i=1}^{p-1} \phi(M_i(x_1,...,x_{p-1},x_{p-1})).$$

Moreover, the quasi-arithmetic mean $K : I^p \to I$, defined by

$$K(x_1,,\ldots,x_p) := \phi^{-1}(\frac{1}{p}\sum_{i=1}^{p} \phi(x_i)), \quad x_1,\ldots,x_p \in I,$$

is $\mathbf{M}$–invariant for the mean-type mapping $\mathbf{M}= (M_1,\ldots,M_p)$.

EXAMPLE 3. Taking $p = 3, I = (0,\infty), \phi(x) = x^2 (x > 0), M_1 = A, M_2 = R$, where A is the arithmetic mean and R is the square-root mean, i.e.

$$A(x,y,z) := \frac{x+y+z}{3}, \quad R(x,y,z) := \left(\frac{x^2+y^2+z^2}{3}\right)^{1/2},$$

it is easy to verify that the conditions (a)-(b) of Proposition 5 are fulfilled. Therefore $M_3 = N$,

$$N(x,y,z) := \frac{1}{3}[3(x^2+y^2+z^2)+(x-y)^2+(y-z)^2+(z-x)^2]^{1/2},$$

is a mean and the mean-type mapping $\mathbf{M}\colon (0,\infty)^3 \to (0,\infty)^3, \mathbf{M} = (A, R, N)$, is K-invariant with $K = R$, i.e.

$$R(A(x,y,z), R(x,y,z), N(x,y,z)) = R(x,y,z), \qquad x,y,z > 0.$$

Moreover, in view of Theorem 1 (or Theorem 2),

$$\lim_{n\to\infty} \mathbf{M}^n = (R,R,R).$$

4. Mean–type mappings and nonexpansivity. Examples

According to Remark 2, every mean-type mapping restricted to the diagonal is the identity map. The identity of I^p is an example of a mean-type mapping which, being an isometry, is of course nonexpansive. The following example is less trivial:

EXAMPLE 4. The map $\mathbf{M}\colon I^p \to I^p$, defined by

$$\mathbf{M}(x_1, x_2, \ldots, x_p) := (x_1, x_1, x_2, \ldots, x_{p-1}), \qquad x_1, \ldots, x_p \in I,$$

is, of course, a nonexpansive (with respect to the Euclidean norm) mean-type mapping, and we have

$$\lim_{n\to\infty} \mathbf{M}^n(x_1,\ldots,x_p) = \mathbf{M}^{p-1}(x_1,\ldots,x_p) = (x_1,x_1,\ldots,x_1).$$

The next example shows that there are mean-type mappings which are neither nonexpansive nor expansive.

EXAMPLE 5. Take $p = 2$ and $I = (0,\infty)$. Then the mean-type mapping $\mathbf{M}\colon (0,\infty)^2 \to (0,\infty)^2$, $\mathbf{M} = (A, G)$, where A and G are, respectively, the arithmetic and geometric mean, is neither nonexpansive nor expansive in the sense of the Euclidean norm. In fact, for $x, y \in (0,\infty)^2$ such that

$$x = (a, a+h), \qquad y = (b, b+h), \qquad a, b, h > 0, \quad a \neq b,$$

we have

$$\mathbf{M}(x) = \left(a + \frac{h}{2}, \sqrt{a(a+h)}\right), \quad \mathbf{M}(y) = \left(b + \frac{h}{2}, \sqrt{b(b+h)}\right),$$

$$\| x - y \|^2 = 2(a-b)^2,$$

$$\| \mathbf{M}(x) - \mathbf{M}(y) \|^2 = 2(a-b)^2 + 2ab + ah + bh - 2\sqrt{ab(a+h)(b+h)},$$

and, since

$$2\sqrt{ab(a+h)(b+h)} < 2ab + ah + bh, \quad a, b, h > 0,$$

(which can be easily verified by taking the second power of both sides) we infer that

$$\| \mathbf{M}(x) - \mathbf{M}(y) \| > \| x - y \| .$$

On the other hand, taking

$$x, y \in (0,\infty)^2, x = (a,b), y = (ta, tb), \qquad a, b, t > 0, t \neq 0,$$

we have

$$\mathbf{M}(x) = \left(\frac{a+b}{2}, \sqrt{ab}\right), \qquad \mathbf{M}(y) = \left(t\frac{a+b}{2}, t\sqrt{ab}\right),$$

$$\| x - y \|^2 = (t-1)^2(a^2+b^2), \qquad \| \mathbf{M}(x) - \mathbf{M}(y) \|^2 = (t-1)^2[(\frac{a+b}{2})^2 + ab],$$

and, clearly,

$$\| \mathbf{M}(x) - \mathbf{M}(y) \| < \| x - y \| .$$

Actually we have shown that M is neither nonexpansive nor expansive in each of the sets $\{x = (a,b) : a, b > 0, a < b\}$ and $\{x = (a,b) : a, b > 0, a > b\}$. Note that $\mathbf{M}(a,b) = \mathbf{M}(b,a)$.

5. Remark on iterative functional equations

In the theory of iterative functional equations (cf. M. Kuczma [6]) a very important role is played by the following

FACT. *Let $I \subseteq \mathbb{R}$ be an interval and $a \in \mathbb{R}$ a point belonging to the closure of I. If $f : I \to \mathbb{R}$ a continuous function such that*

$$0 < \frac{f(x) - a}{x - a} < 1, \qquad x \in I \setminus \{a\}, \tag{7}$$

then $f : I \to I$, and for every $x \in I$,

$$\lim_{n\to\infty} f^n(x) = a.$$

Note that condition (7) can be written in the equivalent form

$$\min(x, a) < f(x) < \max(x, a), \qquad x \in I \setminus \{a\}.$$

This observation leads immediately to the following finite–dimensional counterpart of the above fact (which is easily verified):

REMARK 6. Let $p \in \mathbb{N}$ be fixed. Suppose that $I \subseteq \mathbb{R}$ is an interval and $a \in \mathbb{R}$ a point belonging to the closure of I. If $\mathbf{f} : I^p \to \mathbb{R}^p$, $\mathbf{f} = (f_1, \ldots, f_p)$ is a continuous map such that

$$\min(x_1, \ldots, x_p, a) < f_i(x_1, \ldots, x_p) < \max(x_1, \ldots, x_p, a), \qquad x_i \neq a, \quad i = 1, \ldots, p,$$

then $\mathbf{f}$: $I^p \to I^p$, and for every $x \in I^p$,

$$\lim_{n\to\infty} \mathbf{f}^n(x) = (a, \ldots, a).$$

Acknowledgement. The author is indebted to the referee for his valuable remarks.

References

[1] J. Aczél, *Problem 291*, Aequationes Math. **46** (1993), 199.

[2] J. Borwein, *Problem 291, Solution 1*, Aequationes Math. **47** (1994), 115–118.

[3] P. S. Bullen, D. S. Mitrinović, P. M. Vasić, *Means and their inequalities*, Mathematics and its Applications, D. Reidel Publishing Company, Dordrecht–Boston–Lancaster–Tokyo 1988.

[4] P. Flor, F. Halter-Koch, *Über Folgen, die bezüglich eines Mittels recurrent sind*, Results in Mathematics **26** (1994), 264–273.

[5] P. Kahlig, J. Matkowski, *On the composition of homogeneous quasi arithmetic means*, J. Math. Anal. Appl. **216** (1997), 69–85.

[6] M. Kuczma, *Functional equations in a single variable*, Monografie Matematyczne **46**, PWN – Polish Scientific Publishers, Warszawa 1968.

Intitute of Mathematics
Silesian University
Bankowa 14
PL–40–007 Katowice
Poland

e-mail: matkow@omega.im.wsp.zgora.pl