Title: Differentiable solutions of functional equations in Banach spaces

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Annales Mathematicae Silesianae 7. Katowice 1993, 17–55

Prace Naukowe Uniwersytetu Śląskiego nr 1399

DIFFERENTIABLE SOLUTIONS OF FUNCTIONAL EQUATIONS IN BANACH SPACES

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Abstract. We deal with the functional equation

$$\varphi(F(x)) = g(x, \varphi(x))$$

where functions F and g are given, defined in open subsets of Banach spaces and taking values in Banach spaces as well. We prove theorems on the existtence and uniqueness of solutions of the equation in classes of differentiable functions. As corollaries we get some results on the conjugacy of diffeomorphisms. Analogous results have been known in the finite dimensional case only.

Introduction. The aim of the present paper is to investigate existence and sometimes also uniqueness of local solutions of the functional equation of the first order

$$\varphi(F(x)) = g(x, \varphi(x)),$$

where given functions F and g are defined in open subsets of some Banach spaces and take values in Banach spaces, too.

An inspiration for our work comes from at least two sources. Observe that a particular case of the above written equation is the so called conjugacy equation

$$\varphi(F(x)) = G(\varphi(x)),$$

which, especially when G is linear, plays an important role in the theory of differential equations. When one deals with a dynamic system then solving this equation leads to linearization of the problem. Linearization is discussed in numerous papers under the assumption that given functions are defined in Euclidean spaces. Since we are interested here in looking for solutions

Received revised October 16, 1992.

AMS (1991) subject classification: Primary 39B52. Secondary 47H17.

in higher regularity classes let us mention Ph. Hartman [6], S. Sternberg [27] and [28] and M. Kuczma [19]. These authors solve the linearization equation when the linear operator is a contraction or, more generally, when it is a hyperbolic mapping with the spectrum not intersecting the unit circle.

The present paper also contains results of such a type. However, the given mappings are defined now in spaces of infinite dimension. Thus theorems proved below extend mentioned above (cf. also Remarks 1.1 and 1.2 and comments after Theorem 3.3). Let us recall that a theorem on continuous linearization of a hyperbolic diffeomorphism of a Banach space into itself is known as Grobman–Hartman theorem and is quoted for instance in Z. Nitecki's book [24] (Theorem 2.2).

Another reason to write this paper comes from a natural temptation to generalize some earlier results from the theory of functional equations. Many authors investigated existence and uniqueness of differentiable solutions of equations of the first order. In particular let us mention the papers of B. Choczewski [3], J. Matkowski [22] and [23] and the book of M. Kuczma [18] (Chapter IV).

Our Theorem 1.2 is a generalization of a result contained in [3] to the case of some Banach spaces. Moreover, this theorem says more about the asymptotic properties of solutions at the fixed point of F. Let us note that J. Matkowski's theorems from [12] and [13] have weaker assumptions (m–th derivatives of g need not be Lipschitzian with respect to the second variable) but on the other hand their statements are valid in $\mathbb{R}$ and existence of open relatively compact sets plays a crucial role in proofs. It is interesting that to prove uniqueness of solutions we need not require that derivatives of g are Lipschitzian (cf. Theorem 1.3).

Also Theorems 2.1 and 2.2 concerning the dependence of solutions on an arbitrary function are extensions of well known theorems from [3] and [18] (Chapter IV) to the infinite dimensional case.

As the situation changes when we go from one dimensional case to multidimensional one it seems to be more adequate to compare our results with those obtained by Belitskii [2] or Kučko [9]–[17], cf. also [1] and [8]. Some of the results stated by the former authors are generalized here – not only because the space is more general but also because so are some mappings (cf. Remarks 1.1 and 1.2 and comments after Theorem 3.3).

The paper contains four sections. The first one includes basic notions and some technical lemmas and theorems. They give formulae for higher order derivatives of composite functions in Banach spaces, describe the possibility of extending a function from a neighbourhood of 0 onto the whole space with regularity properties preserved. Also possibility of introducing equivalent norms so that the norms of operators are close to their spectral radius is dealt with.

We use standard notation in our paper. In particular derivatives are denoted as in real case, we use the same symbol $\| \ \|$ for norms in different Banach spaces since it does not lead to any confusion. Similarly 0 stands for zero in different spaces but it is always clear what we mean.

I would like to dedicate this paper to memory of Professor Marek Kuczma (1935-1991) whose inspirating and motivating remarks made this paper be written.

§ **0.** We shall first recall or prove some results that will be useful in the sequel. In what follows the differentiability of mappings will be understood in the sense of Fréchet (see e.g. [4]). If X, Y are Banach spaces, U a nonempty open subset of X and $f : U \to Y$ is k–times differentiable then we use the symbol $f^{(k)}(x)$ for the k–th derivative of the mapping f at the point x and $f^{(k)}(x)(h_1, \dots, h_k)$ denotes the value of $f^{(k)}(x)$ on the vector $(h_1, \dots, h_k) \in X^k$. $C^m(U,Y)$ will mean the family of all mappings $f : U \to Y$ which are m–times continuously differentiable. We write $L^k(X,Y)$ for the set of all k–linear forms from X into Y.

If U and V are nonempty open subsets of Banach spaces X and Y respectively, Z is a Banach space and $g \in C^k(U \times V, Z)$ then for $(x,y) \in U \times V$ the symbol $g^{(k)}_{X^j Y^{k-j}}(x,y)$ means a partial derivative (j–times with respect to the first and $k-j$–times with respect to the second variable). All such partial derivatives are equal independently of the order of differentiation.

In the set $\mathbb{Z}^n$ of all n–tuples of integers define the following relation of partial order: if $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ belong to $\mathbb{Z}^n$ then

$$a \leq b \Leftrightarrow a_i \leq b_i \quad \text{for every} \quad i \in \{1, \dots, n\}.$$

Denote by $\mathbb{Z}^n_+$ all n–tuples of positive integers and for $b = (b_1, \dots, b_n) \in \mathbb{Z}^n$ put $b! := b_1! \dots b_n!$ and $|b| := b_1 + \dots + b_n$.

L. E. Fraenkel in [5] proved the following.

LEMMA 0.1 *Let X, Y, Z be Banach spaces, U and V nonempty open subsets of X and Y, respectively. Let $f \in C^m(U,V)$ and $g \in C^m(V,Z)$ for some $m \in \mathbb{N}$. Then $u := g \circ f \in C^m(U,Z)$ and for every $k \in \{1, \dots, m\}$, $x \in U$ and $h = (h_1, \dots, h_k) \in X^k$ we have*

$$\begin{aligned} &u^{(k)}(x) \\ (0.1) \qquad &= \sum_{n=1}^{k} \sum_{|b|=k} \sum_{\sigma} (n!b!)^{-1} g^{(n)}(f(x))(f^{(b_1)}(x)(h_{\sigma(1)}, \dots, h_{\sigma(b_1)}), \\ &\quad \dots, f^{(b_n)}(x)(h_{\sigma(k-b_n+1)}, \dots, h_{\sigma(k)})) \end{aligned}$$

where $b = (b_1, \dots, b_n)$ *are from* $\mathbb{Z}_+^n$ *and* $\sum_\sigma$ *means the sum over all permutations* σ *of the set* $\{1, \dots, k\}$.

The next lemma can be proved by induction requiring only some skill in computation of derivatives (cf. [18], Chapter IV).

LEMMA 0.2 *Let* U *and* V *be nonempty open subsets of Banach spaces* X *and* Y *respectively, let* Z *be a Banach space,* $g \in C^m(U \times V, Z)$, $\varphi \in C^m(U, V)$ *for an* $m \in \mathbb{N}$. *Then for every* $k \in \{1, \dots, m\}$, $x \in U$ *and* $h = (h_1, \dots, h_k) \in X^k$ *we have*

$$\begin{aligned} g(\cdot, \varphi(\cdot))^{(k)}(x)h = & g'_Y(x, \varphi(x)) \circ \varphi^{(k)}(x)h + P_k(x, \varphi(x), \varphi'(x))h \\ & + R_k(x, \varphi(x), \dots, \varphi^{(k-1)}(x))h. \end{aligned} \tag{0.2}$$

Here

$$P_1(x, \varphi(x), \varphi'(x)h = g'_X(x, \varphi(x))h$$

and

$$P_k(x, \varphi(x), \varphi'(x))h = \sum_{j=0}^{k} \sum_{c_j} g^{(k)}_{X^{k-j}Y^j}(x, \varphi(x)) \circ A^k_{jc_j}(x)h,$$

where $A^k_{jc_j}(x)$ *are cartesian products of* $id_X(k - j - \text{times})$ *and* $\varphi'(x)$ $(j - \text{times})$ *and* $\sum_{c_j}$ *denotes the sum over* $\binom{k}{j}$ *such products.*

Further, for every $x \in U$, $R_k(x, \varphi(x), \dots, \varphi^{(k-1)}(x)) \in L^k(X, Z)$ *can be represented as a finite sum of terms of the form*

$$g^{(r)}_{S_1, \dots, S_r}(x, \varphi(x)) \circ B^k_{S_1, \dots, S_r}(x), \qquad r \in \{2, \dots, k-1\}, \tag{0.3}$$

with $S_j = X$ *or* $S_j = Y$ *for* $j \in \{1, \dots, r\}$ *and* $B^k_{S_1 \dots S_r}(x)$ *being cartesian products of* id_X, $\varphi'(x), \dots, \varphi^{(k-1)}(x)$ *containing at least one derivative of* φ *at* x *of order at least two.*

COROLLARY 0.1 *Under the assumptions of Lemma 0.2, if moreover* $V = Y$ *and* $g(x, \cdot) \in L(Y, Z)$ *for every* $x \in U$ *then for every* $k \in \{1, \dots, m\}$, $x \in U$ *and* $h = (h_1, \dots, h_k) \in X^k$ *we have*

$$g(\cdot, \varphi(\cdot))^{(k)}(x)h = \sum_{j=0}^{k} \sum_{\sigma_j} g^{k-j}_{X^{k-j}}(x, \varphi^{(j)}(x)h^j_{\sigma_j})h^{k-j}_{\sigma_j}, \tag{0.4}$$

where σ_j *is a choice of* j *numbers from the set* $\{1, \dots, k\}$, *coordinates of* $h^j_{\sigma_j} \in X^j$ *are the coordinates of* h *corresponding to this choice and coordinates of* $h^{k-j}_{\sigma_j} \in X^{k-j}$ *are the remaining coordinates of* h.

We also have

COROLLARY 0.2. *Let X and Y be Banach spaces, $\mathbb{K}$ the scalar field of Y and U a nonempty open subset of X. If $\varphi \in C^m(U,Y)$, $\alpha \in C^m(U,\mathbb{K})$ then for every $k \in \{1,\dots,m\}$, $x \in U$, $h = (h_1,\dots,h_k) \in X^k$*

$$(0.5) \qquad (\alpha(\cdot)\varphi(\cdot))^{(k)}(x)h = \sum_{j=0}^{k} \sum_{\sigma_j} (\alpha^{(k-j)}(x)h_{\sigma_j}^{k-j}\varphi^{(j)}(x)h_{\sigma_j}^{j})$$

where σ_j, $h_{\sigma_j}^j$, $h_{\sigma_j}^{k-j}$ have the same meaning as in Corollary 0.2.

Since we will often use equivalent norms in what follows we need the following

LEMMA 0.3. *Let U be a nonempty open subset of a Banach space X and let Y be a Banach space. Then for any mapping $\varphi : U \to Y$ the class of its regularity and the values of derivatives do not depend on equivalent norms. Moreover, if $\| \|_X^1$ and $\| \|_X^2$ are equivalent norms in$X$, $\| \|_Y^1$ and $\| \|_Y^2$ are equivalent norms in Y and $\varphi \in C^m(U,Y)$ then there exist $d > 0$ and $D > 0$ such that*

$$\begin{aligned} d \sup \{\|\varphi^{(m)}(x)\|_1 : x \in U\} &\leq \sup \{\|\varphi^{(m)}(x)\|_2 : x \in U\} \\ &\leq D \sup \{\varphi^{(m)}(x)\|_1 : x \in U\}, \end{aligned}$$

where $\| \|_i$, $i = 1,2$, are norms in $L^m(X,Y)$ generated by $\| \|_X^i$ and $\| \|_Y^i$, $i = 1.2$.

We omit here an immediate proof of this lemma.

Our method of solving functional equations in the sequel will require possibility of extending mappings from a neighbourhood of 0 onto the whole space in such a way that the extension be as regular as the original mapping. The procedure is similar to that used in [6] (Chapter IX). There, however, the problem was solved in the case of finite dimensional spaces.

We shall prove an "extension lemma" for the class of Banach spaces satisfying the following condition

(C) There exists a functional $q \in C^\infty(X,\mathbb{R})$ such that

(i) $$\bigvee_{N\in\mathbb{N}} \bigvee_{C\geq 1} \bigwedge_{x\in X} (C\|x\|^N \geq q(x) \geq \|x\|^N),$$

(ii) $$\bigvee_{D>0} \bigwedge_{1\leq k\leq N} \bigwedge_{x\in X} (\|q^{(k)}(x)\| \leq D\|x\|^{N-k}),$$

(iii) $$\bigwedge_{k>N} (q^{(k)} = 0).$$

REMARK 0.1. (iii) implies by the mean value theorem that $q^{(N)} : X \to L^N(X, \mathbb{R})$ is a constant mapping.

EXAMPLES 1. Every real Hilbert space with the square of the norm as q satisfies (C).

2. Let $p \in \mathbb{N}$. Every space $L^{2p}(\Omega, \Sigma, \mu)$ of real functions integrable with $2p$–th power satisfies (C) with $q = \|\cdot\|^{2p}$. The same holds true for spaces l^{2p} of real sequences summable with $2p$–th power.

REMARK 0.2. In view of Lemma 0.3 it is clear that replacing a norm in X by an equivalent one preserves (C). Moreover, N remains unchanged in (C) and multiplying functional q "good" for the original norm by a suitable constant we get a functional "good" for the new norm.

Before stating next lemma let us quote the following version of the inverse function theorem (cf. [7], [24])

PROPOSITION 0.1. *Let X, Y be Banach spaces, $L \in L(X, Y)$ a bijection and let $\psi : X \to Y$ be a Lipschitz mapping with Lipschitz constant Lip $(\psi) \leq \|L^{-1}\|^{-1}$. Then $L + \psi$ is a bijection and $(L + \psi)^{-1}$ is a Lipschitz mapping with Lip $((L + \psi)^{-1}) \leq 1/(\|L^{-1}\|^{-1} -$ Lip $(\psi))$.*

Now we shall prove the following

LEMMA 0.4. *Let X and Y be Banach spaces and let X satisfy (C). Let U be an open neighbourhood of $0 \in X$. If $F \in C^m(U, Y)$ and $F(0) = 0$ then for every $\varepsilon > 0$ there exists a $\delta' > 0$ such that for every $\delta \in (0, \delta')$ there exists a function $F_{\delta,\varepsilon} \in C^m(X, Y)$ with the following properties*

$$\bigvee_{R>0} (\|x\| < \delta \cdot R \Rightarrow F_{\delta,\varepsilon}(x) = F(x)); \quad \|x\| > \delta \Rightarrow F_{\delta,\varepsilon}(x) = F'(0)x, \tag{0.6}$$

$$\bigwedge_{x \in X} (\|F'_{\delta,\varepsilon}(x)\| \leq \|F'(0)\| + \varepsilon), \tag{0.7}$$

$$\begin{aligned} &\bigvee_{L>0} \bigwedge_{1 \leq r \leq m} \bigwedge_{x \in X} (\|x\| < \delta \Rightarrow \\ &\|(F_{\delta,\varepsilon} - F'(0))^{(r)}(x)\| \leq L \sum_{j=0}^{r} \delta^{j-r} \|(F - F'(0))^{(j)}(x)\|), \end{aligned} \tag{0.8}$$

(0.9) if $F'(0)$ is a bijection then for ε small enough $F_{\delta,\varepsilon}$ is a bijection, too and $\bigwedge_{x\in X}(\|(F_{\delta,\varepsilon}^{-1})'(x) \le 1/(\|F'(0)^{-1}\|^{-1} - \varepsilon))$.

PROOF. For $x \in U$ we can write $F(x) = F'(0)x + \varphi(x)$, where $\varphi \in C^m(U,Y)$, $\varphi(0) = 0 \in Y$, $\varphi'(0) = 0 \in L(X,Y)$. Let q, c and N be as in (C) and put $R := 1/C^{2N}$. Moreover, let $Q := q^{(N)}$. Let $\alpha \in C^\infty(\mathbb{R},\mathbb{R})$ be such a function that

$$\alpha(t) = \begin{cases} 1, & |t| < R, \\ 0, & |t| \ge 1/C. \end{cases}$$

Denote by $M_0 := \sup\{|\alpha^{(k)}(t)| : \ t \in \mathbb{R}, \ k \in \{0,\dots,m\}\}$. Fix an $\varepsilon > 0$ and choose $\delta' > 0$ in such a way that $\|x\| \le \delta' \Rightarrow x \in U$ and

$$\sup\{\|\varphi(x)\| : \|x\| < \delta'\} \le \varepsilon/(\|Q\|M_0 + 1) \tag{0.10}$$

and

$$\sup\{\|F^{(m)}(x) - F^{(m)}(0)\| : \|x\| < \delta'\} < +\infty \tag{0.11}$$

For a $\delta < \delta'$ and $x \in X$ put

$$\varphi_{\delta,\varepsilon}(x) = \begin{cases} \alpha(q(x)/C\delta^N)\varphi(x), & \|x\| < \delta, \\ 0, & \|x\| \ge \delta. \end{cases}$$

It is easy to see that then $F_{\delta,\varepsilon} := F'(0) + \varphi_{\delta,\varepsilon} \in C^m(X,Y)$ and fulfils (0.6). Further, if $\|x\| < \delta$ and $h \in X$ then (cf. Corollary 0.2)

$$\begin{aligned} F'_{\delta,\varepsilon}(x)h = &F'(0)h + (\alpha'(q(x)/C\delta^N)(1/C\delta^N)q'(x)h)\varphi(x) \\ &+ \alpha(q(x)/C\delta^N)\varphi'(x)h \end{aligned}$$

which implies by the choice of δ', (C), the mean-value theorem and (0.10)

$$\begin{aligned} \|F'_{\delta,\varepsilon}(x) - F'(0)\| \le &M_0(1/C\delta^N)\|Q\|\delta^{N-1}\sup\{\|\varphi'(x)\| : \|x\| < \delta\}\cdot\delta \\ &+ \sup\{\|\varphi'(x)\| : \|x\| < \delta\} \le \varepsilon, \end{aligned}$$

and (0.7) follows.

Observe that by the mean-value theorem for $x, y \in X$ we have

$$\begin{aligned} \|\varphi_{\delta,\varepsilon}(x) - \varphi_{\delta,\varepsilon}(y)\| &\le \sup\{\|\varphi_{\delta,\varepsilon}(z)\| : z \in X\}\|x - y\| \\ &= \sup\{\|F'_{\delta,\varepsilon}(z) - F'(0)\| : z \in X\}\|x - y\| \le \varepsilon\|x - y\|, \end{aligned}$$

so the constant Lip $(\varphi_{\delta,\varepsilon}) \leq \varepsilon$. If $F'(0)$ is a bijection and $\varepsilon < \|F'(0)^{-1}\|^{-1}$ then we may use Proposition 0.1 to get (0.9).

In order to prove (0.8) define $\beta : X \to \mathbb{R}$ by $\beta(x) = \alpha(q(x)/C\delta^N)$. Using Corollary 0.2 we obtain for $r \in \{1, \dots, m\}$ and $h = (h_1, \dots, h_r) \in X^r$, if $\|x\| < \delta$,

$$\|(F_{\delta,\varepsilon} - F'(0))^{(r)}(x)h\| = \|\varphi_{\delta,\varepsilon}^{(r)}(x)h\| = \|\sum_{j=0}^{r} \sum_{\sigma_j} (\beta^{(r-j)}(x)h_{\sigma_j}^{r-j})\varphi^{(j)}(x)h_{\sigma_j}^{j}\|.$$

Hence

$$\|\varphi_{\delta,\varepsilon}^{(r)}(x)\| \leq \sum_{j=0}^{r} \binom{r}{j} \|\beta^{(r-j)}(x)\| \cdot \|\varphi^{(j)}(x)\|. \tag{0.12}$$

Now, from Lemma 0.1 we obtain for all $n \in \mathbb{N}$, $h = (h_1, \dots, h_n) \in X^n$ and $x \in X$ such that $\|x\| < \delta$

$$\begin{aligned} &\beta^{(n)}(x)h \\ &= \sum_{k=1}^{n} \sum_{|b|=n} \sum_{\sigma} (k!b!)^{-1}(\alpha^{(k)}(q(x)/C\delta^N))/C^k\delta^{Nk}(q^{(b_1)}(x)h_\sigma^{b_1}, \\ &\quad \dots, q^{(b_k)}(x)h_\sigma^{b_k}) \end{aligned}$$

whence

$$\begin{aligned} &\|\beta^{(n)}(x)\| \\ &\leq \sum_{k=1}^{n} \sum_{|b|=n} n!(b!k!)^{-1}(C\delta^N)^{-k}M_0\|q^{(b_1)}(x)\| \cdot \dots \cdot \|q^{(b_k)}(x)\|. \end{aligned} \tag{0.13}$$

Condition (iii) in (C) implies that on the right hand side of the above inequality only those summands do not vanish for which $b_i \leq N$, $i \in \{1, \dots, k\}$. Putting $I_N = \{1, \dots, N\}$ and using (C) we derive from (0.13)

$$\|\beta^{(n)}(x)\| \leq \sum_{k=1}^{n} \sum_{\substack{b \in I_N^k \\ |b|=n}} n!(b!k!)^{-1}(C\delta^N)^{-k}M_0D^k\delta^{Nk-n}$$

whence

$$\|\beta^{(n)}(x)\| \leq \left[\sum_{k=1}^{n}(M_0n!/k!)(D/C)^k \left(\sum_{\substack{b \in I_N^k \\ |b|=n}} (b!)^{-1}\right)\right] / \delta^n. \tag{0.14}$$

Denote the expression in square brackets by M_n and put

$$L := \max\{\tbinom{r}{j} M_{r-j};\ j \in \{0, \dots, r\}\}.$$

We see that (0.12) and (0.14) imply (0.8) which ends the proof. □

Although looking for local solutions of the functional equations, we will use a method of finding fixed points of some operators defined on subspaces of the space of bounded functions from X into Y. To define the operators properly we will need extensions of some functions defined in a neighbourhood of $0 \in X \times Y$ onto a cylinder in $X \times Y$ containing X. The extensions should preserve regularity of the original functions. This is why we prove the following extension lemma.

LEMMA 0.5. *Let X and Y be Banach spaces and let X satisfy (C). Let U and V be open neighbourhoods of origins in X and Y, respectively. Further, let $G \in C^m(U \times V, Y)$ for a positive integer m and suppose that $G(0,0) = 0$ and $G'_X(0,0) = 0$. Then for every $\varepsilon > 0$ there exist $\delta' > 0$ and $\varrho' > 0$ such that for every $\delta \in (0, \delta')$ there is a mapping $G_{\delta,\varepsilon} \in C^m(X \times \{y \in Y : \|y\| \le \varrho'\}, Y)$ with the following properties*

$$\bigvee_{R>0} [(\|x\| \le \delta \cdot R \wedge \|y\| \le \varrho' \Rightarrow G_{\delta,\varepsilon}(x,y) = G(x,y)) \wedge (\|x\| \ge \delta \ \wedge\ \|y\| \le \varrho' \Rightarrow G_{\delta,\varepsilon}(x,y) = G'_Y(0,0)y)] \tag{0.15}$$

$$\sup\{\|(G_{\delta,\varepsilon})'_Y(x,y) - G'_Y(0,0)\| : (x,y) \in X \times \{y \in Y : \|y\| \le \varrho'\}\} \le \varepsilon, \tag{0.16}$$

$$\sup\{\|(G_{\delta,\varepsilon})'_X(x,y)\| : (x,y) \in X \times \{y \in Y : \|y\| \le \varrho'\}\} \le \varepsilon, \tag{0.17}$$

$$\bigvee_{K>0} \bigwedge_{1 \le r \le m} \bigwedge_{\substack{x \in X \\ y \in Y}} (\|x\| \le \delta \ \wedge\ \|y\| \le \varrho' \Rightarrow \|G^{(r)}_{\delta,\varepsilon}(x,y) - G'_Y(0,0)^{(r)}(y)\| \le K \sum_{j=0}^{r} (\delta^{j-r}) \|G^{(j)}(x,y) - G'_Y(0,0)^{(j)}(y)\|), \tag{0.18}$$

$$\bigvee_{S>0} \bigwedge_{1 \le r \le m} \bigwedge_{\substack{x \in X \\ y \in Y}} (\|x\| \le \delta \ \wedge\ \|y\| \le \varrho' \Rightarrow \|(G_{\delta,\varepsilon})^{(r)}_{X^r}(x,y) - G'_X(0,y)^{(r)}(x)\| \le S \sum_{j=0}^{r} \delta^{(j-r)} \|G^{(r)}_{X^j}(x,y) - G'_X(0,y)^{(j)}(x)\|). \tag{0.19}$$

Moreover, if there exists a $\varrho \in (0, \varrho']$ *such that*

$$\bigvee_{L_1>0} \bigwedge_{x\in U} \bigwedge_{y,\overline{y}\in Y} (\|y\| \le \varrho \ \wedge \ \|\overline{y}\| \le \varrho \Rightarrow \|G^{(m)}(x,y) - G^{(m)}(x,y)\| \le L_1\|y-\overline{y}\|) \tag{0.20}$$

then

$$\bigvee_{L_2>0} \bigwedge_{x\in U} \bigwedge_{y,\overline{y}\in Y} (\|y\| \le \varrho \ \wedge \ \|\overline{y}\| \le \varrho \Rightarrow \|G^{(m)}_{\delta,\varepsilon}(x,y) - G^{(m)}_{\delta,\varepsilon}(x,\overline{y})\| \le L_2\|y-\overline{y}\|(\sum_{j=1}^{m} \delta^{j-m} + \delta^{-m} \sup\{\|G'_Y(x,y)\| : \|x\| \le \delta, \|y\| \le \varrho'\}). \tag{0.21}$$

PROOF. Let q, Q, R, α and M_0 be the same as in the proof of the preceding lemma. Fix $\varepsilon > 0$ and choose ϱ' and δ' in such a way that the following relation holds.

$$\|x\| \le \delta' \ \wedge \ \|y\| \le \varrho' \Rightarrow \|G'_Y(x,y) - G'_Y(0,0)\| + \|G'_X(x,y)\| < \min\{\varepsilon, \varepsilon/2M_0\|Q\|, \varepsilon/M_0\|Q\| + 1\} \tag{0.22}$$

and

$$\bigwedge_{1\le r\le m} s_r := \sup\{\|G^{(r)}(x,y)\| : \|x\| \le \delta', \ \ \|y\| \le \varrho'\} < +\infty. \tag{0.23}$$

Put for $\delta \in (0, \delta')$, all $x \in X$ and $y \in Y$ such that $\|y\| \le \varrho'$

$$\gamma_{\delta,\varepsilon}(x,y) = \begin{cases} \alpha(q(x)/C\delta^N)\gamma(x,y), & \|x\| < \delta, \\ 0, & \|x\| \ge \delta, \end{cases}$$

where $\gamma := G - G'_Y(0,0)$. It is easy to see that $G_{\delta,\varepsilon} : X \times \{y \in Y : \|y\| \le \varrho'\} \to Y$ given by $G_{\delta,\varepsilon}(x,y) = G'_Y(0,0)y + \gamma_{\delta,\varepsilon}(x,y)$ satisfies (0.15); (0.16) and (0.17) are also obvious because by (0.22)

$$\|(G_{\delta,\varepsilon})'_Y(x,y) - G'_Y(0,0)\| \le |\alpha(q(x)/\delta^N C)\ \|\gamma'_Y(x,y)\| \le \|G'_Y(x,y) - G'_Y(0,0)\| \le \varepsilon$$

and

$$\begin{aligned}
\|(G_{\delta,\varepsilon})'_X(x,y)\| &= \|(\gamma_{\delta,\varepsilon})'_X(x,y)\| \\
&\le |\alpha'(q(x)/C\delta^{-N})|\|Q\|\|x\|^{N-1}(C\delta^N)^{-1}\|\gamma(x,y)\| \\
&\quad + \|\gamma'_X(x,y)\| \le M_0\|Q\|\delta^{-1}\sup\{\|\gamma'(x,y)\| : \|x\| \le \delta',\ \|y\| \le \varrho'\}\delta \\
&\quad + \|\gamma'_X(x,y)\| \le (M_0\|Q\| + 1)\sup\{\|\gamma'_Y(x,y)\| \\
&\quad + \|\gamma'_X(x,y)\| : \ \ \|x\| \le \delta' \wedge \|y\| \le \varrho'\} \le \varepsilon
\end{aligned}$$

if $\|x\| \le \delta$ and $\|y\| \le \varrho'$.

An analogous computation as in the proof of Lemma 0.4 gives (0.18). To obtain (0.19) note that putting $F_y(x) := G(x,y)$ for a fixed y (with $\|y\| \le \varrho'$) we define a function F_y which satisfies all assumptions of Lemma 0.4. Moreover, for $x \in U$ and $r \in \{1,\dots,m\}$ we have $F_y^{(r)}(x) = G_{X^r}^{(r)}(x,y)$. From (0.22) it results that δ' defined there is small enough for the inequality (0.10) to hold for F_y. We may then apply Lemma 0.4 for $\delta \le \delta'$ which gives (0.19).

Let us proceed to the proof of the last statement. To this aim put for $(x,y) \in X \times Y$ such that $\|x\| \le \delta$, $\|y\| \le \varrho'$

$$\beta(x,y) = \alpha(q(x)/C\delta^N).$$

Then for all $r \in \{1,\dots,m\}$ and $(h,t) = ((h_1,t_1),\dots,(h_r,t_r)) \in (X \times Y)^r$ we have

$$\beta^{(r)}(x,y)(h,t) = \alpha(q(\cdot)/C\delta^N)^{(r)}(x)h.$$

Applying Corollary 0.2 we get for all $x \in X$ and $y, \overline{y} \in Y$ such that $\|x\| \le \delta$, $\|y\| \le \varrho$ and $\|\overline{y}\| \le \varrho$

$$\begin{aligned}
&\|G_{\delta,\varepsilon}^{(m)}(x,y) - G_{\delta,\varepsilon}^{(m)}(x,\overline{y})\| \\
&\quad = \|\gamma_{\delta,\varepsilon}^{(m)}(x,y) - \gamma_{\delta,\varepsilon}^{(m)}(x,\overline{y})\| \\
&\quad \le \sum_{j-0}^{m} \binom{m}{j} \|\beta^{(m-j)}(x,y)\|\|G^{(j)}(x,y) - G^{(j)}(x,\overline{y})\| \\
&\quad \le \|G^{(m)}(x,y) - G^{(m)}(x,\overline{y})\| + \|y - \overline{y}\| \\
&\qquad \times \sum_{j=0}^{m-1} \binom{m}{j} M_{m-j}\delta^{j-m} \sup\{\|(G^{(j)})'_Y(x,y)\| : \|x\| \le \delta,\ \ \|y\| \le \varrho\} \\
&\quad \le L_2(1 + \sum_{j=1}^{m-1} \delta^{j-m} + \delta^{-m} \sup\{\|G'_Y(x,y)\| : \|x\| \le \delta,\ \|y\| \le \varrho\}\|y - \overline{y}\| \\
&\quad = L_2(\sum_{j=1}^{m} \delta^{j-m} + \delta^{-m} \sup\{\|G'_Y(x,y)\| : \|x\| \le \delta,\ \ \|y\| \le \varrho\}\|y - \overline{y}\|,
\end{aligned}$$

where M_n are defined as in the proof of the preceding lemma and $L_2 := \max\{L_1, \ \max\{\binom{m}{j} M_{m-j} s_j : \ j \in \{1, \dots, m\}\}\}$ (cf. (0.23)). □

Second part of our introductory remarks contains some facts on linear operators we shall use in the next chapter.

For a given operator $A \in L(X, X)$, where X is a Banach space, denote by $\mathrm{sp}(A)$ the spectrum of A, and by $r_s(A)$ the spectral radius of A (equal to $\lim_{n\to\infty} \|A^n\|^{1/n}$; in complex Banach spaces we have $r_s(A) = \sup|\mathrm{sp}\ (A)|$). Note also that the value of the spectral radius does not depend on equivalent norms in X.

Suppose now that X is a real Banach spaces. Then it can be embedded into a complex Banach space $Z = X + iX$ with the norm defined by

$$\|x + iy\| = \sup\{(|x^*(x)|^2 + |x^*(y)|^2)^{1/2} : \ x^* \in X^*, \ \|x^*\| \le 1\},$$

where X^* denotes the dual space of X. It can be easily shown that $A \in L(X, X)$ is a bijection if and only if the operator $\overline{A} : Z \to Z$ defined by

$$\overline{A}(x + iy) = Ax + iAy \tag{0.24}$$

is a bijection. Moreover we have for every $n \in \mathbb{N}$

$$\|A^n\| \le \|\overline{A}^n\| \le 2\|A^n\| \tag{0.25}$$

and also $\overline{A}^{-1}(x + iy) = A^{-1}x + iA^{-1}y$ for every $x + iy \in Z$ if $\overline{A}$ is invertible.

Let us prove the following

LEMMA 0.6. *Let X be a real Banach space and let $A \in L(X, X)$ be a bijection. Then X can be displayed into a direct sum of spaces X_1 and X_2 invariant under A if and only if $Z = X + iX$ can be displayed into a direct sum of spaces Z_1 and Z_2 invariant under $\overline{A}$ defined by (0.24) and the following relations hold*

$$r_s(A|X_1) = r_s(\overline{A}|Z_1) \quad \text{and} \quad r_s((A|X_2)^{-1}) = r_s((\overline{A}|Z_2)^{-1}). \tag{0.26}$$

PROOF. To prove the "if" part put $Z_1 = X_1 + iX_1$ and $Z_2 = X_2 + iX_2$. Both spaces are obviously invariant under $\overline{A}$. From the definition of r_s and (0.25) we obtain the first equality in (0.26). The second one follows by replecing A and $\overline{A}$ by A^{-1} and $\overline{A}^{-1}$ in (0.26). On the other hand let Z be a direct sum of Z_j, $j = 1, 2$. Then there exist real Banach spaces X_j, $j = 1, 2$, such that $Z_j = X_j + iX_j$, $j = 1, 2$. It is not difficult to check that X is a direct sum of X_j, $j = 1, 2$. □

The next lemma will enable us to change the norm in X so that the induced operator norm of a given $A \in L(X,X)$ is close to $r_s(A)$.

LEMMA 0.7. *(cf. [21]). Let $\| \ \|$ be a norm in X and let $A \in L(X,X)$ be a bijection. Then for every $\varepsilon > 0$ there exists a norm $\| \ \|_\varepsilon$ equivalent to $\| \ \|$ and such that*

$$\|A\|_\varepsilon \leq r_s(A) + \varepsilon \quad \text{and} \quad \|A^{-1}\|_\varepsilon \leq r_s(A^{-1}) + \varepsilon.$$

(Here $\|A\|_\varepsilon$ and $\|A^{-1}\|_\varepsilon$ denote the operator norms of A and A^{-1} induced by $\| \ \|_\varepsilon$).

Now let us quote after F. Riesz and B. Sz. – Nagy [26]

THEOREM 0.1. *Let Z be a complex Banach space and let $A \in L(Z,Z)$. Moreover, suppose that sp $(A) = S_1 \cup S_2$ and dist $(S_1, S_2) > 0$ if $S_i \neq \emptyset$, $i = 1,2$. Then Z can be displayed into a direct sum of subspaces Z_i, $i = 1,2$, invariant under A and such that sp $(A|Z_i) = S_i$, $i = 1,2$.*

From the above theorem and preceding lemmas follows

THEOREM 0.2. *Let X be a real Banach space with the norm $\| \ \|$ and let $A \in L(X,X)$ be a bijection. Suppose that sp $(\overline{A}) = S_1 \cup S_2$, where $\overline{A}$ is given by (0.24) and dist $(S_1, S_2) > 0$ if $S_i \neq \emptyset$, $i = 1,2$. Then for every $\varepsilon > 0$ there exists a norm $\| \ \|_\varepsilon$ in X equivalent to $\| \ \|$ and there is a display of X into a direct sum of subspaces X_1 and X_2 invariant under A and such that*

$$\|A|X_1\|_\varepsilon \leq \sup |S_1| + \varepsilon \tag{0.27}$$

and

$$\|(A|X_2)^{-1}\|_\varepsilon \leq (\inf |S_2| - \varepsilon)^{-1}. \tag{0.28}$$

(Here on the left hand sides of the above inequalities are the norms of suitable operators induced by $\| \ \|_\varepsilon$).

PROOF. In view of Theorem 0.1 and Lemma 0.6 we can display X into a direct sum of subspaces X_1 and X_2 invariant under A and such that $r_s(A|X_1) = r_s(\overline{A}|Z_1) = \sup |S_1|$ and $r_s((A|X_2)^{-1}) = r_s((\overline{A}|Z_2)^{-1}) = \inf |S_2|^{-1}$, where Z_i, $i = 1,2$, are as in Theorem 0.1.

Using Lemma 0.7 for $\varepsilon_1 = \min\{\varepsilon, \varepsilon/[\inf |S_2|(\inf |S_2| - \varepsilon)]\}$ we obtain the existence of norms $\| \ \|_i$ in X_i, $i = 1,2$, which are equivalent to norms $\| \ \||X_i$ respectively and such that (0.27) and (0.28) hold with $\| \ \|_\varepsilon$ replaced by $\| \ \|_1$

in (0.27) and by $\| \|_2$ in (0.28). Define for $x = x_1 + x_2 \in X$ a norm $\| \|_\varepsilon$ on putting $\|x\|_\varepsilon = \|x_1\|_1 + \|x_2\|_2$. Then $\| \|_\varepsilon$ is a norm equivalent to $\| \|$ and such that the required inequalities hold. □

REMARK 0.3. Lemma 0.6 and Theorem 0.2 remain valid of course if they are stated for complex Banach spaces – it is sufficient to observe that $X = Z$ and $A = \overline{A}$ in this case.

In the final part of this chapter we shall mention some properties of hyperbolic mappings. Let X be a Banach space. By $\widetilde{X}$ we denote X if it is complex and $X + iX$ if X is real. Similarly for $A \in L(X, X)$ the symbol $\widetilde{A}$ means A if X is complex and $\overline{A}$ if X is real.

For the following definitions cf. [7] and [24].

DEFINITION 0.1. If X is a Banach space then a bijection $A \in L(X, X)$ is called a hyperbolic mapping if sp $(\widetilde{A})$ does not intersect the unit circle.

DEFINITION 0.2. Let U be a neighbourhood of 0 in a Banach space X and let $F : U \to X$ be a diffeomorphism onto $F(U)$. We say that 0 is a hyperbolic fixed point of F if

(i) $F(0) = 0$,

(ii) $F'(0)$ is a hyperbolic mapping.

REMARK 0.4. Theorem 0.2 implies that if A is a hyperbolic mapping in a Banach space X normed with $\| \|$ then there exist a display of X into a direct sum $X_1 + X_2$ of two subspaces invariant under A and a norm $\| \|_1$ equivalent to $\| \|$ and such that norms of $A_1 = A|X_1$ and $A_2 = (A|X_2)^{-1}$ induced by $\| \|_1$ are both less than 1. Conversely, if we assume that such a display and a norm exist for a linear bijection A then A is hyperbolic (cf. [7] and [24]).

Let us conclude the present section with the following

REMARK 0.5 If A is a hyperbolic mapping and I denotes the identity mapping then I–A is a bijection.

§ **1.** In what follows we will deal with the functional equation

$$\varphi(F(x)) = g(x, \varphi(x)). \tag{1.1}$$

Let U be an open neighbourhood of 0 in a Banach space X and let V be an open subset of a Banach space Y. We assume the following hypotheses

(H,1) $F \in C^m(U, X)$ *for an* $m \in \mathbb{N}$, $F(0) = 0$ and $F'(0)$ *is a bijection.*

(H.2) $g \in C^m(U \times V, Y)$.

We will consider the question of existence and uniqueness of solutions of (1.1) in class C^m or its subclasses. Let us start with

DEFINITION 1.1. Every mapping $\varphi_0 \in C^\infty(X,Y)$ which satisfies

$$(1.2)\qquad \begin{aligned} &(\varphi_0 \circ F(\cdot) - g(\cdot, \varphi_0(\cdot)))^{(r)}(0) = 0, \quad r \in \{0, \dots, m\}, \\ &\varphi_0^{(r)}(0) = 0, \quad r > m, \end{aligned}$$

will be called a formal solution of (1.1).

REMARK 1.1. If a sequence $(A_0, A_1, \dots, A_m) \in Y \times L(X,Y) \times \dots \times L^m(X,Y)$ is a solution of (1.2) then $\varphi_0 : X \to Y$ given for $x \in X$ by

$$\varphi_0(x) = \sum_{j=0}^{m} (j!)^{-1} A_j(x, \dots, x)$$

actually is a formal solution of (1.1). On the other hand, solvability of (1.2) is a necessary condition for (1.1) to have a solution defined and of class C^m in a neighbourhood of 0 in X, i. e. a local solution of class C^m.

For any fixed solution φ_0 of (1.1) put $G_{\varphi_0} := g_y(0, \varphi_0(0))$. The present part of our paper deals with the case

$$(A)\qquad \begin{aligned} &\operatorname{sp}(\overline{G_{\varphi_0}}) = S_1 \cup S_2 \quad \text{and} \\ &S_1 \neq \emptyset \Rightarrow \sup |S_1| < \inf |\operatorname{sp}(\overline{F'(0)})|^m, \\ &S_2 \neq \emptyset \Rightarrow \sup |\operatorname{sp}(\overline{F'(0)})|^m < \inf |S_2|, \end{aligned}$$

(cf. (0.24)). Using Theorem 0.2 for G_{φ_0} and Y we can display Y into a sum of subspaces Y_1, Y_2 invariant under G_{φ_0} and introduce a norm $\| \ \|_1$ which is equivalent to the original norm in Y and such that the induced operator norms of $G_{\varphi_0}|Y_i$ satisfy

$$(1.3.1)\qquad \|G_{\varphi_0}|Y_1\|_1 < (\inf |\operatorname{sp}(\overline{F'(0)})|)^m$$

and

$$(1.3.2)\qquad \|(G_{\varphi_0}|Y_2)^{-1}\|_1 < (\sup |\operatorname{sp}(\overline{F'(0)})|)^{-m}.$$

In the case where $S_i = \emptyset$ we have by (0.25) $r_s(G_{\varphi_0}) = \sup |S_j|$, $j \neq i$, and by Lemma 0.7 we may introduce in Y an equivalent norm $\| \ \|_1$ such that (1.3.j) holds with $Y_j = Y$ (observe that $S_1 = \emptyset$ implies that G_{φ_0} is a bijection and

then $r_s(G_{\varphi_0}^{-1}) = (\inf|S_2|)^{-1})$. In what follows we shall deal with the case $S_i \neq \emptyset$, $i = 1,2$, but the remaining case is much simpler and can be easily deduced by neglecting one of the spaces appearing in the reasoning below.

Using Lemma 0.7 for X and $F'(0)$ (which is a bijection in view of **(H.1)**) we may introduce in X an equivalent norm (denote it simply by $\| \ \|$) such that (cf. (1.3.i), i = 1, 2)

$$\|G_{\varphi_0}|Y_1\|_1 < \|F'(0)^{-1}\|^{-m}, \tag{1.4.1}$$

$$\|(G_{\varphi_0}|Y_2)^{-1}\|_1 < \|F'(0)\|^{-m}. \tag{1.4.2}$$

From now on assume that X and Y are normed with $\| \ \|$ and $\| \ \|_1$ respectively, so that (1.3.i) and (1.4.i), $i = 1,2$, hold.

If $\mathcal{U}$ denotes the family of all open neighbourhoods of $0 \in X$ then for $U \in \mathcal{U}$ define the set $\mathcal{A}_U$ by

$$\begin{aligned}\mathcal{A}_U = \{\varphi \in C^m(U,Y) : \ & \varphi^{(j)}(0) = 0, \quad j \in \{0,\dots,m\}, \\ & \sup\{\|\varphi^{(m)}(x)\| : x \in U\} < +\infty\}.\end{aligned}$$

It is clear that $\mathcal{A}_U$ is a Banach space for every $U \in \mathcal{U}$ with the norm given by $\|\varphi\|_U = \sup\{\|\varphi^{(m)}(x)\| : \ x \in U\}$, $\varphi \in \mathcal{A}_U$. Put $\mathcal{A}_{loc} := \bigcup\{\mathcal{A}_U : \ U \in \mathcal{U}\}$.

As we have noticed Y is a direct sum of Y_i, $i = 1,2$. Thus projections $\mathrm{pr}_i : Y \to Y_i$ are of class C^∞ and it can easily be shown that for every $U \in \mathcal{U}$ the space $\mathcal{A}_U$ is a direct sum of the spaces $\mathcal{A}_U^i = \{\varphi \in \mathcal{A}_U : \ \varphi(X) \subset Y_i\}$. Moreover, the norm $\| \ \|$ defined for $\varphi = \varphi_1 + \varphi_2 \in \mathcal{A}_U$ by $\|\varphi\| = \|\varphi_1\|_U + \|\varphi_2\|_U$ is equivalent to $\| \ \|_U$.

For a fixed formal solution φ_0 of (1.1) define the operator $T : C^m(X,Y) \to C^m(U,Y)$ putting for $\varphi \in C^m(X,Y)$

$$T(\varphi) = \varphi \circ F - G_{\varphi_0} \circ \varphi$$

We shall consider also operators $T_{\delta,\varepsilon}$ mapping $\mathcal{A}_X$ into itself and given for $\varphi = \varphi_1 + \varphi_2 \in \mathcal{A}_X$ by

$$T_{\delta,\varepsilon}(\varphi) = (\varphi_1 \circ ((F^{-1})_{\delta,\varepsilon})^{-1} - G_{\varphi_0} \circ \varphi_1) + (\varphi_2 \circ F_{\delta,\varepsilon} - G_{\varphi_0} \circ \varphi_2),$$

where $F_{\delta,\varepsilon}$ and $(F^{-1})_{\delta,\varepsilon}$ are extensions of F and F^{-1}, respectively, defined as in Lemma 0.4. Thus in particular for every $\varepsilon > 0$, suitably chosen $\delta = \delta(\varepsilon) > 0$, $\varphi \in \mathcal{A}_X$ and x from a neighbourhood of $0 \in X$ we have $T(\varphi)(x) = T_{\delta,\varepsilon}(\varphi)(x)$. Assume that G_{φ_0} is a bijection. Our purpose now is to snow that

for ε and δ sufficiently small $T_{\delta,\varepsilon}$ are bijections of $\mathcal{A}_X$ onto itself. To this aim write $T_{\delta,\varepsilon}$ in the following form

$$T_{\delta,\varepsilon}(\varphi) = (I - R_{\delta,\varepsilon}) \circ (\varphi_1 \circ ((F^{-1})_{\delta,\varepsilon})^{-1} + \varphi_2 \circ F_{\delta,\varepsilon}),$$

where I is the identity operator and $R_{\delta,\varepsilon}$ is given by

$$R_{\delta,\varepsilon}(\varphi) = G_{\varphi_0} \circ \varphi_1 \circ (F^{-1})_{\delta,\varepsilon} + G_{\varphi_0} \circ \varphi_2 \circ (F_{\delta,\varepsilon})^{-1}.$$

In view of Remark 0.5 and Lemma 0.4 it is sufficient to show that $R_{\delta,\varepsilon}$ are hyperbolic operators for small ε and δ. This will follow from Remark 0.4. First let us show that $R_{\delta,\varepsilon}|\mathcal{A}^i_X$, $i = 1, 2$, are bijective for small ε and δ. They are invertible since so are G_{φ_0}, $F_{\delta,\varepsilon}$ and $(F^{-1})_{\delta,\varepsilon}$ for small ε and δ. By Lemma 0.1 we have for $r \in \{1, \dots, m\}$, $x \in X$ and $h = (h_1, \dots, h_r) \in X^r$ and $\varphi \in \mathcal{A}^1_X$

$$\begin{aligned}
&(R_{\delta,\varepsilon}(\varphi))^{(r)}(x)h \\
&\quad = G_{\varphi_0} \circ [\varphi^{(r)}((F^{-1})_{\delta,\varepsilon}(x))((F^{-1})'_{\delta,\varepsilon}(x), \dots, (F^{-1})'_{\delta,\varepsilon}(x)) \\
&\qquad + \sum_{n=1}^{r-1} \sum_{|b|=r} \sum_{\sigma} (b!n!)^{-1} \varphi^{(n)}((F^{-1})_{\delta,\varepsilon}(x))((F^{-1})^{(b_1)}_{\delta,\varepsilon}(x), \\
&\qquad \dots, (F^{-1})^{(b_n)}_{\delta,\varepsilon}(x))h,
\end{aligned} \tag{1.5}$$

and in particular

$$(R_{\delta,\varepsilon}(\varphi))^{(r)}(0) = 0, \qquad r \in \{0, \dots, m\}.$$

Further, (1.5) implies for $r = m$

$$\begin{aligned}
\|R_{\delta,\varepsilon}(\varphi)^{(m)}(x)\| &\le \|G_{\varphi_0}|Y_1\|_1 [\|\varphi^{(m)}((F^{-1})_{\delta,\varepsilon}(x))\| \cdot \|(F^{-1})'_{\delta,\varepsilon}(x)\|^m \\
&+ \sum_{n=1}^{m} \sum_{|b|=m} (b!n!)^{-1} m! \|\varphi^{(n)}((F^{-1})_{\delta,\varepsilon}(x)\| \prod_{i=1}^{n} \|(F^{-1})^{(b_i)}_{\delta,\varepsilon}(x)\|].
\end{aligned} \tag{1.6}$$

Take δ so small that (cf. (1.4.1))

$$\|G_{\varphi_0}|Y_1\|_1 (\|(F'(0))^{-1}\| + \varepsilon)^m < 1$$

and denote $\eta_\varepsilon = \|F'(0)^{-1}\| + \varepsilon$ for the sake of brevity. On account of Lemma 0.4 we have for all $x \in X$.

$$\|(F^{-1})'_{\delta,\varepsilon}(x)\| \le \eta_\varepsilon. \tag{1.7}$$

Observe that each summand on the right hand side in (1.6), except for the first one, contains at least one factor of the form $\|(F^{-1})^{(j)}_{\delta,\varepsilon}(x)\|$, $j \geq 2$. According to the definition of $(F^{-1})_{\delta,\varepsilon}$ given in Lemma 0.4 this factor vanishes if $\|x\| \geq \delta$. Thus, because of (1.7), we may write

$$\sup\{\|(R_{\delta,\varepsilon}(\varphi))^{(m)}(x)\| : \ x \in X\} \leq \|G_{\varphi_0}|Y_1\|_1(\eta_\varepsilon^m\|\varphi\|_X + L_\delta),$$

where

$$L_\delta = \sum_{n-1}^{m-1} \sum_{|b|=m} (b!n!)^{-1} m! \sup\{\|\varphi^{(n)}((F^{-1})_{\delta,\varepsilon}(x))\| \cdot \prod_{i=1}^{n} \|(F^{-1})^{(b_i)}_{\delta,\varepsilon}(x)\| : \ \|x\| \leq \delta\}.$$

From the mean value theorem and (1.7) we obtain for all $x \in X$ and $n \in \{1, \dots, m-1\}$

$$\begin{aligned}\|\varphi^{(n)}((F^{-1})_{\delta,\varepsilon}(x))\| &\leq \sup\{\|\varphi^{(m)}(y)\| : \ \|y\| \\ &\leq \|(F^{-1})_{\delta,\varepsilon}(x)\|\}\|(F^{-1})_{\delta,\varepsilon}(x)\|^{m-n} \\ &\leq \|\varphi\|_X(\eta_\varepsilon\|x\|)^{m-n}\end{aligned}$$

whence

$$\sup\{\|\varphi^{(n)}((F^{-1})_{\delta,\varepsilon}(x))\| : \ \|x\| \leq \delta\} \leq \|\varphi\|_X(\eta_\varepsilon\delta)^{m-n}. \tag{1.8}$$

In view of the condition (0.8) from Lemma 0.4 and the mean value theorem we have for $r \in \{2, \dots, m\}$ and $\|x\| \leq \delta$

$$\begin{aligned}\|(F^{-1})^{(r)}_{\delta,\varepsilon}(x)\| &\leq L\sum_{j=0}^{r} \|(F^{-1} - F'(0)^{-1})^{(j)}(x)\|\delta^{j-r} \\ &= L\sum_{j=2}^{r} \|(F^{-1})^{(j)}(x)\|\delta^{j-r} + \|F^{-1}(x) - F'(0)^{-1}(x)\|\delta^{-r} \\ &\quad + \|(F^{-1})'(x) - F'(0)^{-1}\|\delta^{1-r} \leq L\sum_{j=2}^{r} \|(F^{-1})^{(j)}(x)\|\delta^{j-r} \\ &\quad + \sup\{\|(F^{-1})''(y)\| : \ \|y\| \leq \|x\|\}(\|x\| + \delta)\|x\|\delta^{-r}.\end{aligned}$$

Hence we obtain for $r \in \{2, \dots, m\}$ and $\|x\| \leq \delta$

$$\|(F^{-1})^{(r)}_{\delta,\varepsilon}(x)\| \leq C_1 \sum_{j=2}^{r} \delta^{j-r}$$

where $C_1 = L\max\{\sup\{(F^{-1})^{(}j)(x)\| : \ \|x\| \le \delta\} : \ j \in \{2,\dots,m\}\}$, which implies

$$\sup\{\|(F^{-1})^{(r)}_{\delta,\varepsilon}(x)\| : \ \|x\| \le \delta\} \le C_2\delta^{2-r} \tag{1.9}$$

with a suitably chosen constant C_2. Now we are able to estimate L_δ using (1.8) and (1.9)

$$\begin{aligned} L_\delta \le & \sum_{n=1}^{m-1} \sum_{|b|=m} (b!n!)^{-1} m! \|\varphi\|_X \eta_\varepsilon^{m-n} \delta^{m-n} \eta_\varepsilon^{p_n} C_2^{n-p_n} \delta^{2n-m-p_n} \\ \le & C_3 \sum_{n=1}^{m} \delta^{n-p_n} \|\varphi\|_X, \end{aligned}$$

where C_3 is a constant and p_n denotes the number of "ones" in the sequence $(b_1,\dots,b_n)$. Of course $p_n < n$ for $n \in \{1,\dots,m-1\}$ since $b_1+\dots+b_n = m$. Hence a real number D exists such that

$$L_\delta \le D\delta\|\varphi\|_X, \tag{1.10}$$

and therefore

$$\sup\{\|(R_{\delta,\varepsilon}(\varphi))^{(m)}(x)\| : \ x \in X\} \le (\|G_{\varphi_0}|Y_1\|_1 \eta_\varepsilon^m + D\delta)\|\varphi\|_X.$$

Choosing ε and δ sufficiently small we get

$$\|R_{\delta,\varepsilon}(\varphi)\|_X \le q_{\delta,\varepsilon}\|\varphi\|_X \tag{1.11}$$

for a $q_{\delta,\varepsilon} \in (0,1)$ and

$$\lim_{(\delta,\varepsilon)\to(0,0)} q_{\delta,\varepsilon} = \|G_{\varphi_0}|Y_1\|_1 \|F'(0)^{-1}\|^m < 1. \tag{1.12}$$

This proves in particular that $R_{\delta,\varepsilon}$ are contractions of $\mathcal{A}^1_X$ for δ and ε sufficiently small. Similarly as above we can obtain surjectivity of $R_{\delta,\varepsilon} \mid \mathcal{A}^1_X$. In an analogous way we establish that for δ and ε sufficiently small $R_{\delta,\varepsilon} \mid \mathcal{A}^2_X$ are bijective and for all $\varphi \in \mathcal{A}^2_X$

$$\|(R_{\delta,\varepsilon})^{-1}(\varphi)\|_X \le p_{\delta,\varepsilon}\|\varphi\|_X \tag{1.13}$$

for a $p_{\delta,\varepsilon} \in (0,1)$ with

$$\lim_{(\delta,\varepsilon)\to(0,0)} p_{\delta,\varepsilon} = \|(G_{\varphi_0}|Y_2)^{-1}\|_1 \|F'(0)\|^m < 1. \tag{1.14}$$

Thus $T_{\delta,\varepsilon}$ are bijections for δ and ε small enough. Putting $r_{\delta,\varepsilon} := \max\{q_{\delta,\varepsilon}, p_{\delta,\varepsilon}\}$ we obtain

$$\|T_{\delta,\varepsilon}^{-1}\| \leq 1/\min\{1 - r_{\delta,\varepsilon}, r_{\delta,\varepsilon}^{-1} - 1\} \tag{1.15}$$

in view of the inequalities

$$\|T_{\delta,\varepsilon}^{-1}\varphi\|_X \leq \|(I - R_{\delta,\varepsilon})^{-1}\|(\|\varphi_1\|_X + \|\varphi_2\|_X) = \|(I - R_{\delta,\varepsilon})^{-1}\| \cdot \|\varphi\|$$

and

$$\begin{aligned}\|(I - R_{\delta,\varepsilon})\varphi\| =& \|(I - R_{\delta,\varepsilon})\varphi_1\|_X + \|(I - R_{\delta,\varepsilon})\varphi_2\|_X \\ \geq& \|\varphi_1\|_X - \|R_{\delta,\varepsilon}\varphi_1\|_X + \|R_{\delta,\varepsilon}\varphi_2\|_X - \|\varphi_2\|_X \\ \geq& (1 - r_{\delta,\varepsilon})\|\varphi_1\|_X + (r_{\delta,\varepsilon}^{-1} - 1)\|\varphi_2\|_X \\ \geq& \min\{1 - r_{\delta,\varepsilon}, r_{\delta,\varepsilon}^{-1} - 1\}\|\varphi\|.\end{aligned}$$

(1.12) and (1.14) imply

$$\lim_{(\delta,\varepsilon)\to(0,0)} \|T_{\delta,\varepsilon}^{-1}\|_X < +\infty. \tag{1.16}$$

Using similar arguments one can show that for δ and ε sufficiently small operators $T_{\delta,\varepsilon}$ are self–bijections of the space

$$\begin{aligned}\mathcal{C} =& \{\varphi \in C^{m-1}(X,Y) : \ \varphi^{(r)}(0) = 0, \quad r \in \{1, \dots, m-1\}, \\ & \sup\{\|\varphi^{(m-1)}(x)\|/\|x\| : \ x \in X\} < +\infty\}\end{aligned}$$

endowed with the norm $\|\varphi\|_{\mathcal{C}} = \sup\{\|\varphi^{(m-1)}(x)\|/\|x\| : \ x \in X\}$ and

$$\lim_{(\delta,\varepsilon)\to(0,0)} \|T_{\delta,\varepsilon}^{-1}\| < +\infty. \tag{1.17}$$

Taking into account (1.16) and (1.17) we may choose $M < +\infty$, $\varepsilon_1 > 0$ and $\delta_1 > 0$ in such a way that

$$\sup\{\max\{\|T_{\delta,\varepsilon}^{-1}\|, \|T_{\delta,\varepsilon}^{-1}\|_{\mathcal{C}}\} : \ \delta < \delta_1, \ \varepsilon < \varepsilon_1\} \leq M. \tag{1.18}$$

It is obvious that $\mathcal{A}_X \subset \mathcal{C}$ by the mean value theorem.

Define also

$$\begin{aligned}\mathcal{B} =& \{\varphi \in C^{m-1}(X,Y) : \varphi^{(r)}(0) = 0, r \in \{1, \dots, m-1\}, \mathrm{Lip}(\varphi^{(m-1)}) \\ & < +\infty\}.\end{aligned}$$

We have of course $\mathcal{A}_X \subset \mathcal{B}$. Note also that if $\varphi \in \mathcal{A}_X$ then $\|\varphi\|_c \leq \|\varphi\|_X$. We will say that a $\varphi : X \to Y$ is a local solution of (1.1) if $\varphi|U$ satisfies (1.1) for a neighbourhood of $0 \in X$.

After these remarks let us formulate the following

THEOREM 1.1. *Let U be an open neighbourhood of 0 in a Banach space X satisfying (C) and let V be an open nonempty subset of a Banach space Y and assume that (H.1) and (H.2) are fulfilled. If φ_0 is a formal solution of (1.1) such that (A) holds then (1.1) has a local solution $\psi \in \mathcal{B} + \varphi_0$. If moreover g satisfies*

$$\|g^{(m)}(x,y) - g^{(m)}(x,\overline{y})\| \leq L\|y - \overline{y}\| \tag{1.19}$$

for an $L < +\infty$ and all $x \in U$ and $y, \overline{y}$ belonging to neighbourhood of $\varphi_0(0)$ then (1.1) has a local solution $\psi \in \mathcal{A}_X + \varphi_0$.

PROOF. Let φ_0 be a formal solution of (1.1) for which (A) holds. It follows from the definition that $\varphi_0(0) \in V$. Hence by continuity of φ_0 there exist $c > 0$ and $d > 0$ such that $V' + \varphi_0(U') \subset V$ for $V' = K(0,d) \subset V$ and $U' = K(0,c) \subset U$. Thus for $(x,y) \in U' \times V'$ we can define $G \in C^m(U' \times V', Y)$ by

$$G(x,y) = g(x, y + \varphi_0(x)) - g(x, \varphi_0(x)) - G_{\varphi_0} y.$$

Define also $\tau \in C^m(U', Y)$ putting for $x \in U'$

$$\tau(x) = g(x, \varphi_0(x)) - \varphi_0(F(x)).$$

Suppose now that for some $\delta > 0$, $\varepsilon > 0$ and $\varrho' > 0$, $T_{\delta,\varepsilon}$ is a modification of T defined above and $\tau_{\delta,\varepsilon} \in C^m(X,Y)$, $G_{\delta,\varepsilon} \in C^m(X \times K(0,\varrho'), Y)$ are extensions of τ and G respectively, as described in Lemmas 0.4 and 0.5. Consider the equation

$$T_{\delta,\varepsilon}(\varphi)(x) - G_{\delta,\varepsilon}(x, \varphi(x)) = \tau_{\delta,\varepsilon}(x) \tag{1.20}$$

for $x \in X$ and $\varphi : X \to K(0,\varrho')$. It is easy to observe that if φ is a solution of (1.20) then $\varphi + \varphi_0$ is a local solution of (1.1). Thus our proof is reduced to finding a solution of (1.20) belonging to $\mathcal{B}$ or $\mathcal{A}_X$, respectively.

Let $\varepsilon \in (0, \min\{\varepsilon_1, /2M + 1\})$ and $\delta_2 \in (0, \delta_1)$ (cf. (1.18)) be such that $T_{\delta_2,\varepsilon}$ is at the same time a continuous bijection of $\mathcal{C}$ onto $\mathcal{C}$ and $\mathcal{A}_X$ onto $\mathcal{A}_X$. Further, as by the definition of φ_0 all derivatives of τ vanish at 0 we may choose by Lemma 0.4 a $\delta_3 \in (0, \delta_1)$ and an $L_1 > 0$ such that

$$\sup\{\|\tau^{(m)}(x)\| : \|x\| \leq \delta_3\} < \varepsilon/m.$$

and (cf. (0.8))

$$\|\tau_{\delta,\varepsilon}\|_{\mathcal{C}} \leq \|\tau_{\delta_3,\varepsilon}\|_X \leq L_1\varepsilon. \tag{1.21}$$

Observe that

$$G(x,0) = 0 \quad \text{for} \quad x \in U' \quad \text{and} \quad G'_Y(0,0) = 0. \tag{1.22}$$

This implies that

$$G^{(r)}_{X^r}(x,0) = 0 \tag{1.23}$$

for $x \in U'$ and $r \in \{1,\dots,m\}$. Moreover, by Lemma 0.2 we have for every $\varphi \in C^r(U',V')$

$$G(\cdot,\varphi(\cdot))^{(i)}(0) = 0, \qquad i \in \{0,\dots,r\}, \tag{1.24}$$

if $\varphi^{(i)}(0) = 0 \;\; i \in \{0,\dots,r\}$.

Let $R = ML_1\varepsilon/(1-2M\varepsilon)$ and let $\delta' > 0$ and $\varrho' > 0$ be as in Lemma 0.5. Choose a $\delta_4 \in (0,\min\{\delta',\delta_1\})$ in such a way that $R\delta_4^{m-1} < \min\{1,\varrho'/\delta_4\}$. Then, if $\varphi \in \mathcal{C}$ and $\|\varphi\|_{\mathcal{C}} \leq R$, it follows by the mean value theorem that $\|\varphi(x)\| \leq \min\{\delta_4,\varrho'\}$ for all $\|x\| \leq \delta_4$. Since G is of class C^m and because of (1.23) we may take δ_4 so small that for $r \in \{0,\dots,m\}$, $j \in \{0,\dots,r\}$

$$\sup\{\|G^{(r)}_{X^jY^{r-j}}(x,y)\| : \; \|x\| \leq \delta_4, \;\; \|y\| \leq \delta_4\} =: N < +\infty, \tag{1.25}$$

and

$$\sup\{\|G^{(m)}_{X^m}(x,y)\| : \; \|x\| \leq \delta_4, \;\; \|y\| \leq \delta_4\} < \varepsilon. \tag{1.26}$$

Let $G_{\delta_4,\varepsilon}$ be a modification of G described in Lemma 0.5. Taking into account (0.16) and (1.22) we get

$$\sup\{\|(G_{\delta_4,\varepsilon})'_Y(x,y)\| : \;\; x \in X, \;\; \|y\| \leq \varrho'\} \leq \varepsilon. \tag{1.27}$$

In view of (1.20) we get

$$G_{\delta_4,\varepsilon}(\cdot,\varphi(\cdot))^{(r)}(0) = 0, \qquad r \in \{0,\dots,m-1\} \tag{1.28}$$

for every $\varphi \in \mathcal{C}$, $\|\varphi\|_{\mathcal{C}} \leq R$. Using several times the mean value theorem, Lemma 0.5, (1.25), (1.26), (1.27) and the fact that $G_{\delta_4,\varepsilon}(x,y) = 0$ for $\|x\| > \delta_4$, one can show (we omit here the detailed, not difficult computation) that there exists a constant C such that for every $\varphi \in \mathcal{C}$, if $\|\varphi\|_{\mathcal{C}} \leq R$ then

$$\|G_{\delta_4,\varepsilon}(\cdot,\varphi(\cdot))\|_{\mathcal{C}} \leq (\varepsilon + C\delta_4)R. \tag{1.29}$$

Similar arguments lead to the inequality

$$\|G_{\delta_4,\varepsilon}(\cdot,\varphi(\cdot)) - G_{\delta_4,\varepsilon}(\cdot,\psi(\cdot))\|_{\mathcal{C}} \le (\varepsilon + C_1\delta_4)\|\varphi - \psi\|_{\mathcal{C}} \tag{1.30}$$

for a constant C_1 and all $\varphi, \psi \in \mathcal{C}$, for which $\|\varphi\| \le R$ and $\|\psi\| \le R$. Taking, if necessary, ε and δ_4 smaller we can assume that

$$c\delta_4 < \varepsilon \quad \text{and} \quad M(\varepsilon + C_1\delta_4) < 1. \tag{1.31}$$

Similarly we can obtain that (cf. (1.24))

$$G_{\delta_4,\varepsilon}(\cdot,\varphi(\cdot)) \in \mathcal{A}_X \quad \text{and} \quad \|G_{\delta_4,\varepsilon}(\cdot,\varphi(\cdot))\|_X \le 2\varepsilon R \tag{1.32}$$

for every $\varphi \in \mathcal{A}_X$ with $\|\varphi\|_X \le R$.

Put $\delta = \min\{\delta_2, \delta_3, \delta_4\}$. Then $T_{\delta,\varepsilon}, \tau_{\delta,\varepsilon}$ and $G_{\delta,\varepsilon}$ have all the properties listed above. In particular, solving (1.20) in $\mathcal{C}$ is equivalent to finding fixed points of the map $S : K_R = \{\varphi \in \mathcal{C} : \ \|\varphi\|_{\mathcal{C}} \le R\} \to \mathcal{C}$ defined by

$$S(\varphi) = T_{\delta,\varepsilon}^{-1}(\tau_{\delta,\varepsilon}) + T_{\delta,\varepsilon}^{-1}(G(\cdot,\varphi(\cdot))) \tag{1.33}$$

for $\varphi \in \mathcal{C}$, $\|\varphi\|_{\mathcal{C}} \le R$. On account of (1.18), (1.21), (1.29), and (1.31)

$$\|S(\varphi)\|_{\mathcal{C}} \le \|T_{\delta,\varepsilon}^{-1}\| \cdot \|\tau_{\delta,\varepsilon}\|_{\mathcal{C}} + \|T_{\delta,\varepsilon}^{-1}\| \cdot \|G_{\delta,\varepsilon}(\cdot,\varphi(\cdot))\|_{\mathcal{C}} \le M(L_1\varepsilon + 2\varepsilon R) = R.$$

This implies that $S(K_R) \subset K_R$ since from the definition of $\tau_{\delta,\varepsilon}$ and (1.24) it follows that $S(\varphi)^r((0) = 0$ for $r \in \{0, \dots, m-1\}$. But K_R is a complete metric space with the metric induced by $\|\cdot\|_{\mathcal{C}}$. Moreover, from (1.30) and (1.31) we easily infer that S is a contraction, hence there is a unique $\widetilde{\varphi} \in K_R$ such that $S(\widetilde{\varphi}) = \widetilde{\varphi}$. From the Banach contraction theorem it follows in particular that $\lim_{n\to\infty} \|\widetilde{\varphi} - \varphi_n\|_{\mathcal{C}} = 0$, where $(\varphi_n)_{n\in\mathbb{N}}$ is defined by

$$\varphi_0 = 0, \quad \varphi_{n+1} = S(\varphi_n), \qquad n \in \mathbb{N} \cup \{0\}.$$

By (1.32) we get $\varphi_n \in \mathcal{A}_X \cap K_R$, $n \in \mathbb{N} \cup \{0\}$. Now, fix $x, y \in X$, $x \ne y$. Then for every $n \in \mathbb{N} \cup \{0\}$ we have

$$\begin{aligned}
\|x-y\|^{-1}\|\widetilde{\varphi}^{(m-1)}(y) - \widetilde{\varphi}^{(m-1)}(y)\| \le{}& \|x\|^{-1}\widetilde{\varphi}^{(m-1)}(x) \\
&- \varphi_n^{(m-1)}(x)\| \cdot \|x-y\|^{-1}\|x\| \\
&+ \|y\|^{-1}\|\widetilde{\varphi}^{(m-1)}(y) - \varphi_n^{(m-1)}(y)\| \cdot \|x-y\|^{-1}\|y\| \\
&+ \|x-y\|^{-1}\|\varphi_n^{(m-1)}(x) - \varphi_n^{(m-1)}(y)\| \\
\le{}& \|\widetilde{\varphi} - \varphi_n\|_{\mathcal{C}}\|x-y\|^{-1}(\|x\| + \|y\|) + \|\varphi_n\|_X \\
\le{}& \|\widetilde{\varphi} - \varphi_n\|_{\mathcal{C}}\|x-y\|^{-1}(\|x\| + \|y\|) + R.
\end{aligned}$$

Hence, letting $n \to +\infty$ we obtain for all $x, y \in X$, $x \neq y$,

$$\|x - y\|^{-1} \|\widetilde{\varphi}^{(m-1)}(x) - \widetilde{\varphi}^{(m-1)}(y)\| \leq R.$$

Thus Lip $(\widetilde{\varphi}^{(m-1)}) \leq R$, which means in particular that $\widetilde{\varphi} \in \mathcal{B}$. For $\|x\| \leq \delta/2$ we have $\tau_{\delta,\varepsilon}(x) = \tau(x)$, $T_{\delta,\varepsilon}(\widetilde{\varphi})(x) = T(\widetilde{\varphi})(x)$, $G_{\delta,\varepsilon}(x, \widetilde{\varphi}(x)) = G(x, \widetilde{\varphi}(x))$. This means that $\widetilde{\varphi} + \varphi_0$ satisfies (1.1) for $\|x\| \leq \delta/2$.

Now let us suppose that g satisfies (1.19) for $y, \overline{y} \in \{z \in Y : \|z - \varphi_0(0)\| < \varrho\}$. Then, using Lemma 0.2 and the mean value theorem we can obtain that there exist $\eta > 0$ and $L' < +\infty$ such that

$$\|G^{(m)}(x, y) - G^{(m)}(x, \overline{y})\| \leq L' \|y - \overline{y}\|$$

whenever $\|x\| \leq \eta$, $\|y\| \leq \eta$ and $\|\overline{y}\| \leq \eta$. Using Lemma 0.5 we conclude that for sufficiently small δ and ε a modification $G_{\delta,\varepsilon}$ of G satisfies (0.21). This makes it possible to show that S defined by (1.33) actually is a contraction of $\mathcal{A}_X \cap K_R$ (using (0.21) we are able to obtain (1.30) for $\| \ \|_X$). Thus in this case $\widetilde{\varphi} = S(\widetilde{\varphi}) \in \mathcal{A}_X$ and $\widetilde{\varphi} + \varphi_0$ is a local solution of (1.1) in $\mathcal{A}_X$. □

In some cases we can omit a somehow restricting condition (C) imposed on the space X. It was used above to extend functions onto the whole space X in the same class of regularity, which in turn made the use of Banach theorem possible. The same can be realized if we assume for instance that F or F^{-1} is a topological contraction. First let us prove

THEOREM 1.2. *Let (H.1) and (H.2) be fulfilled and assume that for some* $\beta \geq 0$

$$\|(F^{-1})^{(m)}(x) - (F^{-1})^{(m)}(0)\| = O(\|x\|^{\beta}), \qquad x \to 0. \tag{1.34}$$

Moreover, assume that there is a neighbourhood W of $0 \in X$ such that for every neighbourhood $V \subset W$ of $0 \in X$ there exists a neighbourhood $V' \subset V$ of $0 \in X$ such that $F^{-1}(V') \subset V$. Let φ_0 be a formal solution of (1.1) such that

$$\sup |\operatorname{sp}(\overline{G_{\varphi_0}})| < \inf |\operatorname{sp}(\overline{F'(0)})|^{m+\beta} \tag{1.35}$$

and

$$\|g^{(m)}(x, y) - g^{(m)}(0, \varphi_0(0))\| = O(\|x\|^{\beta} + \|y\|^{\beta}), \quad (x, y) \to (0, \varphi_0(0)). \tag{1.36}$$

Let g satisfy (1.19) for $x \in U$ and $y, \overline{y}$ from a neighbourhood of $\varphi_0(0)$. Then there is exactly one local solution[1] *$\varphi \in \mathcal{A}_{\mathrm{loc}} + \varphi_0$ which satisfies*

$$\|\varphi^{(m)}(x) - \varphi^{(m)}(0)\| = O(\|x\|^{\beta}), \qquad x \to 0. \tag{1.37}$$

[1] i.e. every solution of (1.1) with given properties has to coincide with φ on a neighbourhood of 0.

PROOF. In view of (1.35) and Lemma 0.7 we may assume without loss of generality that norms in X and Y are such that the generated operator norms fulfil

$$\|G_{\varphi_0}\| \cdot \|(F^{-1})'(0)\|^{m+\beta} < 1. \tag{1.38}$$

Denote by $\mathcal{A}_{W,R}$ the set $\{\varphi \in \mathcal{A}_W : \sup\{\|x\|^{-\beta}\|\varphi^{(m)}(x)\| : x \in W\} \leq R\}$. $\mathcal{A}_{W,R'}$ endowed with the metric $d(\varphi,\psi) = \sup\{\|x\|^{-\beta}\|\varphi^{(m)}(x) - \psi^{(m)}(x)\| : x \in W\}$ is a complete metric space for all $W \in \mathcal{U}$ and $R > 0$. Let $R > 0$ be given and let a neighbourhood $W \subset U$ of $0 \in X$ be such that $(\varphi + \varphi_0) \circ F^{-1}(W) \subset V$, for every $\varphi \in \mathcal{A}_{W,R}$ (by the mean value theorem such a choice of W is possible). Then we can define an operator S on $\mathcal{A}_{W,R}$ putting

$$S(\varphi) = g(\cdot, (\varphi + \varphi_0)(\cdot)) \circ F^{-1} - \varphi_0. \tag{1.39}$$

We will show that for some R and suitably chosen W the operator S is a contraction of $\mathcal{A}_{W,R}$. First note that for every $\varphi \in \mathcal{A}_{W,R}$ and $r \in \{0, \dots, m\}$.

$$(S(\varphi))^{(r)}(0) = 0. \tag{1.40}$$

Indeed, $S(\varphi)(0) = g(0, \varphi_0(0)) - \varphi_0(0) = 0$ since φ_0 is a formal solution of (1.1). Suppose that (1.40) holds for $r \in \{0, \dots, k-1\}$ where $k \leq m$. Then $(S(\varphi) \circ F)^{(k)}(0) = (g(\cdot, (\varphi_0 + \varphi)(\cdot)) - \varphi_0 \circ F)^{(k)}(0) = 0$ by the definition of φ_0, Lemma 0.2 and because derivatives of φ vanish at 0. On the other hand, using Lemma 0.1 for $S(\varphi) \circ F$ and the induction hypothesis we obtain

$$0 = (S(\varphi) \circ F)^{(k)}(0) = (S(\varphi))^{(k)}(0)(F'(0), \dots, F'(0)).$$

which ends the induction in view of the invertibility of $F'(0)$.

Let $c_1 \in (0,1)$ and $\varrho > 0$ be such that (cf. (1.38))

$$\begin{aligned} &\sup\{\|g'_Y(x,y)\| : \|x\| \leq c_1, \ \|y\| \leq \varrho\} \cdot \sup\{\|(F^{-1})'(x)\|^{m+\beta} : \|x\| \leq c_1\} \\ &\quad \leq \Theta < 1. \end{aligned} \tag{1.41}$$

From Lemmas 0.1 and 0.2 it follows that for $\varphi \in \mathcal{A}_{W,R}$ if only $\varphi(F^{-1}(x)) + \varphi_0(F^{-1}(x)) \in V$ then we may write

$$\begin{aligned} &\|x\|^{-\beta}\|S(\varphi)^{(m)}(x)\| \\ &\quad \leq \|x\|^{-\beta}\|g'_Y(F^{-1}(x), (\varphi + \varphi_0)(F^{-1}(x)\| \\ &\qquad \cdot \|(F^{-1})'(x)\|^m \|\varphi^{(m)}(F^{-1})(x)\| \\ &\qquad + \|x\|^{-\beta} P(\|\varphi'(F^{-1}(x))\|, \dots, \|\varphi^{(m-1)}(F^{-1}(x))\|) + A(x). \end{aligned}$$

Here P is a polynomial in $m-1$ variables with coefficients being products of norms of partial derivatives if g taken at $(F^{-1}(x),\ (\varphi+\varphi_0)(F^{-1}(x)))$ and derivatives of F^{-1} taken at x, and $A(x)$ is the norm of the sum of remaining terms which do not depend on any derivative $\varphi^{(r)}(F^{-1}(x)),\ r\in\{1,\dots,m\}$. Now, the regularity of given functions, (1.36) and assumptions on φ imply that c_1 and ϱ may be chosen in such a way that for every $c\le c_1$, every $V'\subset\{x\in X:\ \|x\|<c\}$ such that $F^{-1}(V')\subset V'$, every $x\in V'$ and every $\varphi\in\mathcal{A}_{V',R'}$ provided $\|(\varphi+\varphi_0)(F^{-1}(x))-\varphi_0(0)\|\le\varrho$, we will have

$$\|x\|^{-\beta}\|S(\varphi)^{(m)}(x)\|\le\Theta R+M_1cR\overline{P}(R)+M_1, \tag{1.42}$$

where M_1 a constant which does not depend on R and $\overline{P}(R)$ is a polynomial. If we take $R>M_1/1-\Theta$, say $R=M_1/1-\Theta-\Theta_1$ for $\Theta_1\in(0,1-\Theta)$, then taking c sufficiently small we can make sure that $\Theta R+cM_1\overline{P}(R)R+M_1\le R$. Moreover, c may be chosen in such a way that $\|(\varphi+\varphi_0)(F^{-1}(x))-\varphi_0(0)\|\le\varrho$ if $x\in V'$ and $\varphi\in\mathcal{A}_{V',R'}$, what follows from the mean value theorem. Thus taking upper bound of the left hand side of (1.42) we obtain $S(\mathcal{A}_{V',R})\subset\mathcal{A}_{V',R'}$.

It is also possible to show that S actually is a contraction of $\mathcal{A}_{V',R}$ if $V'\subset\{x\in X:\ \|x\|\le c\}$, $F^{-1}(V')\subset V'$ and c is small enough. We would like to omit here a somehow tedious computation of this fact and underline only that it is a consequence of (1.41), (1.19) and the mean value theorem. Thus S has a fixed point φ in $\mathcal{A}_{V',R'}$. Of course $\psi=\varphi+\varphi_0$ is a local solution of (1.1) which satisfies (1.37). If $\psi_1=\varphi_1+\varphi_0$ is another such solution, then there is a neighbourhood of zero $V''\subset V'$ such that $\varphi_1\in\mathcal{A}_{V'',R'}$. Moreover, we can assume that S is a contraction of $\mathcal{A}_{V'',R'}$. Hence S has a unique fixed point in $\mathcal{A}_{V'',R'}$ which implies $\varphi_1|V''=\varphi|V''$. □

Condition (1.19) has been used above to make possible the aplication of Banach theorem. However, it turns out that we can obtain uniqueness without assuming (1.19). In fact we have

THEOREM 1.3. *Let all assumptions of Theorem 1.2 be fulfilled except that g satisfies (1.19). Then there is at most one solution $\varphi\in\mathcal{A}_{\mathrm{loc}}+\varphi_0$ of (1.1) fulfilling (1.37).*

PROOF. Suppose that $\varphi_i,\ i=1,2$, are two local solutions of (1.1) fulfilling (1.37). Introducing equivalent norms if necessary we can fix $\varepsilon>0$ so that

$$q:=(\|G_{\varphi_0}\|+\varepsilon)(\|F'(0)^{-1}\|+\varepsilon)^{m+\beta}<1. \tag{1.43}$$

Let δ_1 and ϱ_1 be such positive numbers that

$$\|g'_Y(x,y)\|\le\|G_{\varphi_0}\|+\varepsilon, \tag{1.44}$$

and

$$\|F^{-1}(x)\| \leq (\|F'(0)^{-1}\| + \varepsilon)\|x\|, \tag{1.45}$$

if $\|x\| \leq \delta_1$ and $\|y\| \leq \varrho_1$. Taking into account the continuity of φ_i, $i = 1, 2$, let $\delta_2 > 0$ be such that

$$\|\varphi_i(x) - \varphi_i(0)\| = \|\varphi_i(x) - \varphi_0(0)\| < \varrho_1, \quad i = 1, 2, \tag{1.46}$$

if $\|x\| < \delta_2$. Moreover (1.37) implies the existence of positive numbers C and δ_3 such that

$$\|\varphi_1(x) - \varphi_2(x)\| \leq C\|x\|^{m+\beta} \tag{1.47}$$

if $\|x\| \leq \delta_3$. Put $\delta := \min\{\delta_1, \delta_2, \delta_3\}$ and take a neighbourhood of zero $\overline{V} \subset \{x \in X : \|x\| < \delta\}$ such that $F^{-1}(\overline{V}) \subset \overline{V}$. Then for $x \in \overline{V}$ we obtain, using (1.43) – (1.47) and induction

$$\begin{aligned}
&\|\varphi_1(x) - \varphi_2(x)\| \\
&\quad = \|g(F^{-1}(x),\ \varphi_1(F^{-1}(x))) - g(F^{-1}(x),\ \varphi_2(F^{-1}(x)))\| \\
&\quad \leq (\|G_{\varphi_0}\| + \varepsilon)\|\varphi_1(F^{-1}(x)) - \varphi_2(F^{-1}(x))\| \\
&\quad \leq \ldots \leq (\|G_{\varphi_0}\| + \varepsilon)^n \|\varphi_1(F^{-n}(x)) - \varphi_2(F^{-n}(x))\| \\
&\quad \leq (\|G_{\varphi_0}\| + \varepsilon)^n C((\|F'(0)^{-1}\| + \varepsilon)^n \|x\|)^{m+\beta} \\
&\quad = q^n C\|x\|^{m+\beta}
\end{aligned}$$

for every $n \in \mathbb{N}$. Hence $\varphi_1(x) = \varphi_2(x)$ for every $x \in \overline{V}$, which ends the proof. □

From Theorem 1.2 we infer

COROLLARY 1.1. *Let F fulfil assumptions of Theorem 1.2. If*

$$\sup|\operatorname{sp}(\overline{F'(0)})| < \sup|\operatorname{sp}(\overline{F'(0)})|^{m+\beta}$$

then for every formal solution φ_0 of the equation

$$\varphi(F(x)) = F'(0)\varphi(x) \tag{1.48}$$

there exists a unique solution $\varphi \in \mathcal{A}_{\mathrm{loc}} + \varphi_0$ of this equation which satisfies (1.37).

PROOF. Put $Y = X$ and define $g : X \times Y \to Y$ by $g(x, y) = F'(0)y$ for $(x, y) \in X \times Y$. Then we have all assumptions of Theorem 1.2 satisfied by

$g|U \times V$, with $V = Y = X$, since $G_{\varphi_0} = F'(0)$ for every formal solution φ_0 of (1.48). □

REMARK 1.1. Observe that local invertibility of F implies that (1.48) is equivalent in neighbourhood of 0 to the equation

$$\varphi(F^{-1}(x)) = F'(0)^{-1}\varphi(x). \tag{1.49}$$

Denoting $f = F^{-1}$ and $S = F'(0)^{-1}$ we can observe that if X is finite dimensional then assumptions of M. Kuczma's Theorem 2 from [19] are the same as the above Corollary (except for those which guarantee the existence of a formal solution of (1.49)). Thus our Corollary extends Kuczma's result on arbitrary Banach spaces. In the same sense Theorem 1.1 extends Belitskii's Corollary 2 from [2].

Suppose now that F and H are diffeomorphisms defined in a neighbourhood of their common fixed point $0 \in X$ and that $F'(0)$ and $H'(0)$ are bijections of X onto itself. Let us recall that F and H are said to be conjugate in class C^m if there exists a local diffeomorphism φ of class C^m solving the equation

$$\varphi(F(x)) = H(\varphi(x)). \tag{1.50}$$

F and H are said to be formally conjugate if there exists a formal solution φ_0 of (1.50) with $\varphi_0(0) = 0$ and $\varphi'_0(0)$ being bijective. It is not difficult to check that formal conjugacy of F and H implies sp $(\overline{F'(0)}) =$ sp $(\overline{H'(0)})$. Now we able to derive from Theorem 1.1 the following

COROLLARY 1.2. *Let F and H be local diffeomorphisms which are defined and of class C^m in a neighbourhood of 0 in a Banach space X satisfying (C). Assume that F and H are formally conjugate, (A) is fulfilled with $G_{\varphi_0} = F'(0)$ and Lip $(H^{(m)}) < +\infty$. Then F and H are conjugate in class C^m.*

PROOF. It is sufficient to observe that (1.50) is a particular case of (1.1). Namely, if we put $g(x,y) = H(y)$ for $(x,y) \in U \times V$ we see that $G_{\varphi_0} = H'(0)$ and hence sp $(\overline{G_{\varphi_0}}) =$ sp $(\overline{F'(0)})$ for every formal conjugacy φ_0. All assumptions of Theorem 1.1 can now be easily verified and existence of a solution of (1.50) follows. As its first derivative at 0 equals $\varphi'_0(0)$ we infer that the solution is a local diffeomorphism. □

REMARK 1.2. The above Corollary 1.2 becomes Corollary 1 from [2] in the case of finite dimensional spaces.

Consider the equation

$$\varphi(x) = h(x, \varphi(f(x))), \tag{1.51}$$

If f is locally invertible then (1.51) is locally equivalent to (1.1) with $F = f^{-1}$ and g given by $g(x,y) = h(f^{-1}(x), y)$. Equation (1.51) was dealt with by B. Choczewski in [3] (cf. also monographies [18] and [20]) without invertibility assumption but only for functions defined on the real line. If $f'(0) \neq 0$ then theorems on existence and uniqueness of solutions of (1.51) proved in the above mentioned paper follow from our Theorem 1.2. Choczewski's theorem was later improved by J. Matkowski (cf. [22] and [23]) who used however compactness of neighbourhoods in finite dimensional spaces to prove his results.

§ **2.** Let us now pass to theorems concerned with non–uniqueness of solutions of (1.1). Further on we shall assume that

$$\sup|\operatorname{sp}(\overline{F'(0)})| < 1. \tag{2.1}$$

Let us prove first

THEOREM 2.1 *Let (H.1), (H.2) and (2.1) be fulfilled. Then there exists a neighbourhood W of $0 \in X$ such that for every function*

$$\psi_0 \in C^m(\operatorname{cl} W \setminus \operatorname{Int} F(\operatorname{cl} W), Y)$$

which satisfies for every $r \in \{0, \dots, m\}$ and $y \in F(\operatorname{cl} W \setminus W)$

$$(\psi_0 - g(\cdot, \psi_0(\cdot)) \circ F^{-1})^{(r)}(y) = 0^2 \tag{2.2}$$

exactly one function exists in $C^m(\operatorname{cl} W \setminus \{0\}, Y)$ which solves (1.1) and its restriction to $\operatorname{cl} W \setminus \operatorname{Int} F(\operatorname{cl} W)$ is equal to ψ_0.

PROOF. From (2.1) we infer that taking, if necessary, an equivalent norm in X we may assume that $\|F'(0)\| \leq \Theta < 1$. Hence we may choose $R > 0$ and $q \in (0,1)$ in such a way that $\|x\| \leq R$ implies $\|F(x)\| \leq q\|x\|$. Moreover, R can be such that F is a diffeomorphism in a ball containing $W := \{x \in X : \|x\| \leq R\}$. Define $(W_n)_{n \in \mathbb{N}_0}$ by $W_0 = W$ and $W_n = F^n(W_0)$ for $n \in \mathbb{N}$. Define also a sequence $(V_n)_{n \in \mathbb{N}_0}$ by $V_0 = \operatorname{cl} W_0 \setminus \operatorname{cl} W_1$, $V_n = F^n(V_0)$ for $n \in \mathbb{N}$. It is clear that

$$\bigcup_{n \in \mathbb{N}_0} V_n = \operatorname{cl} W_0 \setminus \{0\}.$$

[2] i.e. derivatives $\psi_0^{(r)}$, $r \in \{0, \dots, m\}$ are continuous on the boundary of $\operatorname{cl} W \setminus \operatorname{Int} F(\operatorname{cl} W)$ and satisfy (2.2).

Let ψ_0 be as in our assumptions and define $\psi_n : V_n \to Y$ putting

$$\psi_n(x) = g(F^{-1}(x),\ \psi_{n-1}(F^{-1}(x))) \quad \text{for} \quad x \in V_n.$$

Let $\psi :$ cl $W \setminus \{0\} \to Y$ be defined by $\psi|V_n = \psi_n$, $n \in \mathbb{N}_0$. ψ is well defined since F is diffeomorphic in W and thus $V_n \cap V_{n+1} = \emptyset$ for $n \in \mathbb{N}$. Moreover, ψ solves (1.1) in cl $W \setminus \{0\}$. To show the regularity of ψ observe that $F^n(\text{Int } W_0) =$ Int V_n. Hence and from the definition od ψ we get $\psi \in C^m(\bigcup_{n\in\mathbb{N}_0} \text{Int } V_n, Y)$ in view of the regularity of ψ_0. Now, fix a $y \in S = F(\{x \in X : \|x\| = R\}) \cap V_1$. Then $\|y\| \le q\|F^{-1}(y)\|$. Hence a neighbourhood W' of $F^{-1}(y)$ exists which is disjoint with W_1. $F(W')$ is a neighbourhood of y and $F(W') \cap W_2 = F(W') \cap F(W_1) = F(W' \cap W_1) = \emptyset$. Thus, when taking a sequence $(y_n)_{n\in\mathbb{N}}$ convergent to y, we may assume that $\{y_n : n \in \mathbb{N}\} \subset W_0 \setminus$ cl $W_2 \subset V_0 \cup V_1$. Let $(y_{k_n})_{n\in\mathbb{N}}$ be a subsequence of $(y_n)_{n\in\mathbb{N}}$ which is contained in V_0. Then by (2.2) and continuity of ψ_0 we obtain

$$\begin{aligned}\lim_{n\to\infty} \psi(y_{k_n}) &= \lim_{n\to\infty} \psi_0(y_{k_n})\\ &=\psi_0(y) = g(F^{-1}(y),\ \psi_0(F^{-1}(y)))\\ &=\psi_1(y) = \psi(y).\end{aligned}$$

On the other hand, if $(y_{m_n})_{n\in\mathbb{N}}$ is a subsequence of $(y_n)_{n\in\mathbb{N}}$ contained in V_1 then the continuity of given functions and (2.2) imply

$$\begin{aligned}\lim_{n\to\infty} \psi(y_{m_n}) &= \lim_{n\to\infty} \psi_1(y_{m_n})\\ &= \lim_{n\to\infty} g(F^{-1}(y_{m_n}),\ \psi_0(F^{-1}(y_{m_n})))\\ &=g(F^{-1}(y),\ \psi_0(F^{-1}(y))) = \psi(y).\end{aligned}$$

Hence the continuity of ψ in S follows. Now, if $y + h \in V_0$ then

$$\psi(y+h) - \psi(y) = \psi_0(y+h) - \psi_0(y) = \psi'(y)h + o(\|h\|), \qquad h \to 0;$$

and if $y + h \in V_1$ then

$$\begin{aligned}\psi(y+h) =&\psi(y) = \psi_1(y+h) - \psi_1(y)\\ =&g(F^{-1}(y+h),\ \psi_0(F^{-1}(y+h))) - g(F^{-1}(y),\ \psi_0(F^{-1}(y)))\\ =&(g(\cdot, \psi_0(\cdot)) \circ F^{-1})'(y) + o(\|h\|), \qquad h \to 0,\end{aligned}$$

in view of (2.2). Hence it follows that ψ is of class C^1 on S. By induction it is easy to show that ψ is of class C^m on S, and hence on $V_0 \cup V_1$. Using induction again one obtains that $\psi \in C^m(\bigcup_{n\in\mathbb{N}_0} V_n, Y) = C^m(W \setminus \{0\}, Y)$. □

The above theorem will be used to show the following one.

THEOREM 2.2. *Assume (H.1), (H.2) and (2.1). Let φ_0 be a formal solution of (1.1) which satisfies*

$$\sup |\operatorname{sp}(\overline{G_{\varphi_0}})| < \inf |\operatorname{sp}(\overline{F'(0)})|^m.$$

Then there exist a neighbourhood W of $0 \in X$ and a positive number d_0 such that every function $\psi_0 \in C^m(\operatorname{cl} W \setminus \operatorname{Int} F(\operatorname{cl} W), Y)$ which satisfies (2.2) for all $r \in \{0, \dots, m\}$ and $y \in F(\operatorname{cl} W \setminus W)$, and such that

$$\sup \{\|\psi_0(x) - \psi_0(0)\| : \ x \in \operatorname{cl} W \setminus \operatorname{Int} F(\operatorname{cl} W)\} \le d_0 \tag{2.3}$$

and

$$\begin{aligned} \sup \{\|\psi_0^{(r)}(x) - \psi_0^{(r)}(0)\| : \ & x \in \operatorname{cl} W \setminus \operatorname{Int} F(\operatorname{cl} W), \\ & r \in \{1, \dots, m\}\} =: D < +\infty \end{aligned} \tag{2.4}$$

has a unique extension to a solution ψ of (1.1) which is defined and of class C^m in W and $\psi^{(r)}(0) = \psi_0^{(r)}(0)$ for $r \in \{0, \dots, m\}$.

PROOF. Introducing, if necessary, equivalent norms in X and Y we may assume that (1.38) holds with $\beta = 0$ and $\|F'(0)\| < 1$ in view of (2.1). Hence we infer that for all $r \in \{0, \dots, m\}$

$$\|G_{\varphi_0}\| \cdot \|F'(0)^{-1}\|^r < 1. \tag{2.5}$$

Let R be as in the proof of Theorem 2.1 and choose $\delta_0 \le R$ and $d_0 > 0$ in such a way that for all $r \in \{0, \dots, m\}$

$$\|g'_Y(x, y)\| \cdot \|F'(x)^{-1}\|^r \le q < 1 \tag{2.6}$$

and

$$\|g(x, \varphi_0(0)) - g(0, \varphi_0(0))\| \le (1 - q) d_0 \tag{2.7}$$

if $\|x\| \le \delta_0$ and $\|y - \varphi_0(0)\| \le d_0$. Let $D > 0$ be arbitrary. For a $\delta \le \delta_0$ let $W_\delta = \{x \in X : \ \|x\| < \delta\}$ and let $\varphi \in C^m(W_\delta \setminus \{0\}, Y)$ be a solution of (1.1) which satisfies (2.3) and (2.4), such a solution exists in view of Theorem 2.1. From (2.6), (2.7) and the mean value theorem it follows that for all $x \in W_\delta \setminus \operatorname{Int} F(\operatorname{cl} W_\delta)$

$$\begin{aligned} &\varphi(F(x)) - \varphi_0(0)\| \\ &\quad = \|g(x, \varphi(x)) - g(0, \varphi_0(0))\| \\ &\quad \le \|g(x, \varphi(x)) - g(x, \varphi_0(0))\| + \|g(x, \varphi_0(0)) - g(0, \varphi_0(0))\| \\ &\quad \le q\|\varphi_0(x) - \varphi_0(0)\| + (1 - q) d_0 \le d_0. \end{aligned}$$

An easy induction gives for every $n \in \mathbb{N}$

$$
\begin{aligned}
&\|\varphi(F^n(x)) - \varphi_0(0)\| \\
(2.8) \qquad &\leq q^n \|\varphi_0(x) - \varphi_0(0)\| + \sum_{i=0}^{n-1} q^i \|g(F^{n-1-i}(x), \varphi_0(0)) - g(0, \varphi_0(0))\| \\
&\leq q^n d_0 + (1-q)^{-1}(1-q^n) d_0 = d_0.
\end{aligned}
$$

Hence $\|\varphi(x) - \varphi_0(0)\| \leq d_0$ for all $x \in W_\delta \setminus \{0\}$. Moreover, if $\varepsilon > 0$ is fixed then there exist $\delta' > 0$ and $n_0 \in \mathbb{N}$ such that

$$
(2.9) \qquad \|g(z, \varphi_0(0)) - g(0, \varphi_0(0))\| \leq ((1-q)/2)\varepsilon
$$

if $\|z\| < \delta'$ and for $n \geq n_0$

$$
(2.10) \qquad q^n d_0 < \varepsilon/2.
$$

Thus for all $x \in F^n(\{x \in X : \|x\| < \delta'\})$, $n \geq n_0$ we have by (2.8), (2.9) and (2.10)

$$
\|\varphi(x) - \varphi_0(0)\| \leq q^n d_0 + (1-q)^{-1}((1-q)/2)\varepsilon = \varepsilon,
$$

whence $\lim_{x \to 0} \varphi(x) = \varphi_0(0)$. Thus, if $\psi : W_\delta \to Y$ is defined by $\psi(x) = \varphi(x)$ for $x \neq 0$ and $\psi(0) = \varphi_0(0)$ then ψ is a solution of (1.1), continuous at 0 and of class C^m in $W_\delta \setminus \{0\}$. Now, using Lemma 0.1 and Lemma 0.2 we may write for $r \in \{1, \dots, m\}$ and $x \in W_\delta \setminus \{0\}$

$$
\begin{aligned}
&\|\psi^{(r)}(F(x)) - \psi_0^{(r)}(0)\| \\
(2.11) \qquad &\leq \|g'_Y(x, (x))\| \cdot \|\psi^{(r)}(x) - \psi_0^{(r)}(0)\| \cdot \|F'(x)^{-1}\|^r + S_r(x),
\end{aligned}
$$

where $S_r : W_\delta \to \mathbb{R}_+$ is a function continuous at 0 for $r \in \{1, \dots, m\}$ and $\lim_{x \to 0} S_r(x) = 0$, which follows from the assumptions on given functions and just proved continuity of ψ. Let $\delta_1 \leq \delta_0$ be such that

$$
(2.12) \qquad S_r(x) \leq (1-q)D
$$

if $\|x\| \leq \delta_1$ and put $W = W_{\delta_1}$. If ψ_0 satisfies assumptions of our theorem then in view of the above part of the proof it has a unique extension to a solution $\psi : W \to Y$ which is continuous in W and of class C^m in $W \setminus \{0\}$. The proof of the fact that $\lim_{x \to 0} \psi^{(r)}(x) = \varphi_0^{(r)}(0)$ is now a simple repeating

of the argument used in the proof of continuity of ψ. Of course, now (2.11) and (2.12) should be used together with regularity of given functions. □

The above theorem is a generalization of an analogous result of B. Choczewski (cf. [3], [18] and [20], Chapter 5) to the case of infinite dimensional spaces.

§ **3.** The last section of the present paper is devoted to a case in a sense "symmetrical" to the one considered in §1 and §2. Namely we will assume now that absolute values of the spectrum of $\overline{G_{\varphi_0}}$ belong to an interval and absolute values of the spectrum of $\overline{F'(0)}$ are outside this interval.

Let us first quote a theorem on invariant manifolds which is proved in a more general setting in [7] (cf. also [24]) and its finite dimensional version is to be found for instance in Ph. Hartman's book [6] (Chapter IX).

Let X be a Banach space and let 0 be a hyperbolic fixed point of a diffeomorphism F defined in a neighbourhood of 0 (cf. Definition 0.2). Then X may be displayed into a direct sum of subspaces invariant under $F'(0)$ (cf. Remark 0.4). Denote by $X_1(X_2)$ the direct summand corresponding to the part of sp $(\overline{F'(0)})$ lying inside (outside) the unit circle. For a $d > 0$ denote $X(d) = \{x \in X : \|x\| \leq d\}$, $X_i(d) = X(d) \cap X_i$, $i = 1, 2$. Now we can quote the announced theorem.

THEOREM 3.1. *There exists a $d > 0$ such that the functional equation*

$$g((\pi_1 \circ F)(u, g(u))) = \pi_2(F(u, g(u))), \tag{3.1}$$

where π_i, $i = 1, 2$, are projections on X_i, $i = 1, 2$, has exactly one solution $g : X_1(d) \to X_2$ which is nonexpansive and such that $g(0) = 0$. If, moreover, F is of class C^m in $X(d)$ then so is g in $X_1(d)$ and $g'(0) = 0$.

Note that the last equality in the above theorem may be derived from its proof but is not contained in the proof of our next theorem. This proof will be omitted here since it is a simple reproduction of the one given in finite dimensional case in [6] (Theorem 5.1, Ch. IX).

THEOREM 3.2. *If 0 is a hyperbolic fixed point of a local diffeomorphism F then there exist a $d > 0$ and an invertible mapping $R : X(d) \to X$ such that $R(0) = 0$, $R'(0)$ is a bijection and*

$$R \circ F^{-1} \circ R^{-1}(X_2(d)) \subset X_2 \tag{3.2}$$

and

$$R \circ F \circ R^{-1}(X_1(d)) \subset X_1. \tag{3.3}$$

Moreover, if F is of class C^m then so is R.

Now let us proceed to the formulation of the last theorem in our paper. Assumming that

$$1 \notin |\operatorname{sp}(\overline{F'(0)})| \tag{3.4}$$

we will look for local solutions of (1.1) given a formal solution φ_0 fulfilling

$$\begin{aligned}(\sup\{|\operatorname{sp}(\overline{F'(0)})| \cap (0,1)\})^m < \inf |\operatorname{sp}(\overline{G_{\varphi_0}})| \leq \sup(\overline{G_{\varphi_0}})| \\ \leq (\inf\{|\operatorname{sp}(\overline{F'(0)})| \cap (1,+\infty)\})^m.\end{aligned} \tag{B}$$

In order to shorten the statement denote for a fixed φ_0, $M_1 := \sup |\operatorname{sp}(\overline{G_{\varphi_0}})|$, $M_2 := \inf |\operatorname{sp}(\overline{F'(0)})|$, $M_3 := \inf\{|\operatorname{sp}(\overline{F'(0)})| \cap (1,+\infty)\}$. We have the following ($E(t)$ means the entire part of $t \in \mathbb{R}$)

THEOREM 3.3. *Let (H.1), (H.2) and (C) be fulfilled. If φ_0 is a formal solution of (1.1) such that (B) holds and g satisfies (1.19) with a constant L for all $x \in U$ and $y, \overline{y}$ from a neighbourhood of $\varphi_0(0)$ then there exists a local solution φ of (1.1) which is of class C^s where $s = E(m\ln M_3 - \ln M_1/\ln M_3 - \ln M_2) - 1$ and for every $r \in \{0, \dots, s\}$*

$$\varphi^{(r)}(0) = \varphi_0^{(r)}(0). \tag{3.5}$$

PROOF. Let $d > 0$ and $R \in C^m(X(d), X)$ be as in Theorem 3.2. For $x \in X(d)$ write (1.1) in the form

$$(\varphi \circ R^{-1})((R \circ F \circ R^{-1})(R(x))) = g(R^{-1}(R(x)), (\varphi \circ R^{-1})(R(x))).$$

Substituting $z = R(x)$ and denoting $T = R \circ F \circ R^{-1}$ and $\psi = \varphi \circ R^{-1}$ we obtain for $z \in W := R(X(d))$ the equation

$$\psi(T(z)) = g(R^{-1}(z), \psi(z)). \tag{3.6}$$

It is straightforward matter to check that $T \in C^m(W, X)$, $\operatorname{sp}(\overline{T'(0)}) = \operatorname{sp}(\overline{F'(0)})$ since $R'(0)$ is a bijection and $g \circ (R^{-1}, \operatorname{id}_V) \in C^m(W \times V, V)$. Another simple observation is that $\psi_0 := \varphi_0 \circ R^{-1}$ is a formal solution of (3.6) satisfying

$$(g \circ (R^{-1}, \operatorname{id}_V))'_Y(0, \psi_0(0)) = g'_Y(R^{-1}(0), \varphi_0(0)) = G_{\varphi_0}.$$

Moreover, using Lemmas 0.1 and 0.2 and the inclusion $R^{-1}(W) = X(d) \subset U$ we obtain that $g \circ (R^{-1}, \mathrm{id}_V)$ fulfils (1.19) with a constant L_1 and for all $z \in W$ and $y, \overline{y}$ from a neighbourhood of $\psi_0(0) = \varphi_0(0)$. Local invariance of X_i, $i = 1.2$, under T implies in particular

$$T'(0) \mid X_i = (T|X_i)'(0), \quad i = 1,2, \quad \text{and} \quad T'(0)(X_i) \subset X_i, \quad i = 1,2.$$

The first equality for $i = 1$ and the left inequality in (B) allow us to verify that assumptions of Theorem 1.1, with $S_1 = \emptyset$, are fulfilled. Therefore a function $\psi_1^* \in C^m(X_1, Y)$ exists which is a solution of (3.6) in $X_1 \cap W'$, where $W' \subset W$ is a neighbourhood of $0 \in X$. Extend ψ_1^* to a $\psi_1 \in C^m(X, Y)$, putting for $z = z_1 + z_2 \in X$

$$\psi_1(z) = \psi_1^*(z_1) - \psi_0(z_1, 0) + \psi_0(z).$$

In the following we will look for a local solution of (3.6) in the form $\psi = \psi_1 + \psi_2$, where ψ_2 is supposed to be defined and of class C^s in a neighbourhood W_1 of $0 \in X$ and for all $z \in W_1 \cap X$, and $r \in \{0, \dots, s\}$

$$\psi_2^{(r)}(z) = 0. \tag{3.7}$$

By the definition of s, taking into account suitable lemmas from Section 0, we may choose $\varepsilon > 0$ in such a way that the inequality

$$(\|G_{\varphi_0}\| + \varepsilon)(\|T'(0)^{-1}\| + \varepsilon)^s(\|(T'(0)|X_2)^{-1}\| + \varepsilon)^{m-s} < 1 \tag{3.8}$$

holds, after introducing equivalent norms, if necessary. From (3.6) we get the following equation for ψ_2:

$$\psi_2(T(z)) = G(z, \psi_2(z)). \tag{3.9}$$

Here T and G given by $G(z, y) = g(R^{-1}(z), y + \psi_1(z)) - \psi_1(T(z))$ are defined for $z \in \{z \in X : \|z\| \le c_1\}$ and $y \in \{y \in Y : \|y\| \le d_1\}$ with $c_1 < 1$, d_1 chosen in such a way that $y + \psi_1(z) \in V$. If, moreover, $\{z \in X : \|z\| \le c_1\} \subset W'$ then we have for every $r \in \{0, \dots, m\}$

$$G_{X^r}^{(r)}(z, 0) = 0 \tag{3.10}$$

if $\|z\| < c_1$ and $z \in X_1$, which easily follows from the fact that ψ_1 is a solution of (3.6) in $X_1 \cap W'$. We have also

$$G_Y'(0, 0) = G_{\varphi_0}. \tag{3.11}$$

By (3.10) there exist $D > 0$ and $c_2 \in (0, c_1]$ such that for every $r \in \{0, \dots, m\}$

$$\|G^{(r)}_{X^r}(z,0)\| \le D\|z_2\|^{m-r}, \tag{3.12}$$

if only $\|z\| = \|z_1 + z_2\| < c_2$. This is a simple consequence of Taylor theorem.

Diminishing ε in (3.8), if necessary, and taking into account Lemmas 0.4 and 0.5 we may choose a $\delta \in (0, c_2)$ and define functions $G_{\delta,\varepsilon}$ and $(T^{-1})_{\delta,\varepsilon}$ in such a way that $(T^{-1})_{\delta,\varepsilon}$ is invertible and $T^{-1}(X_2(\delta)) \subset X_2$, since X_2 is locally invariant under T. If $\|z\| \le \delta$ then obviously

$$(T^{-1})_{\delta,\varepsilon}(z) = (T^{-1})'(0)z + \alpha(q(z)/C\delta^N)(T^{-1}(z) - (T^{-1})'(0)z),$$

(cf. Lemma 0.4). (0.6) and invariance of X_i, $i = 1, 2$, under $(T^{-1})'(0)$ give

$$(T^{-1})_{\delta,\varepsilon}(X_i) \subset X_i, \quad i = 1, 2.$$

This in turn enables us to conclude by the mean value theorem and Lemma 0.4 that for $z = z_1 + z_2 \in X$

$$\|(\pi_2 \circ (T^{-1})_{\delta,\varepsilon})'(z)\| \le (\|(T'(0)|X_2)^{-1}\| + \varepsilon)\|z_2\| \tag{3.13}$$

since $\|(\pi_2 \circ (T^{-1})_{\delta,\varepsilon})'(z) - (T'(0)|X_2)^{-1}\| = \|\pi_2((T^{-1})'_{\delta,\varepsilon}(z) - T'(0)^{-1})\| \le \|(T^{-1})'_{\delta,\varepsilon}(z) \dot{-} (T^{-1})'(0)\| < \varepsilon$, for every $z \in X$.

From the definition of $G_{\delta,\varepsilon}$ it follows in particular that $G_{\delta,\varepsilon}(z,0) = \alpha(q(z)/C\delta^N)G(z,0)$ for $z \in X$, whence by (3.10) and Corollary 0.2 we obtain that $\Gamma = G_{\delta,\varepsilon}(\cdot, 0)$ satisfies (3.7) for all $z \in X_1$ (we have $\Gamma(z) = 0$ if $\|z\| \ge \delta$). Using the mean value theorem again we derive the following inequality

$$\|\Gamma^{(s)}(z)\| \le \sup\{\|\Gamma^{(m)}(x)\| : \|x\| \le \delta\}\|z_2\|^{m-s}$$

for every $z = z_1 + z_2 \in X$ (note that $\Gamma|X_1 = 0$). From Lemma 0.6 ((0.19)) and (3.12) we have also for $z \in X$

$$\|\Gamma^{(m)}(z)\| \le S\sum_{j=0}^{m} \delta^{j-m}\|G^{(j)}_{X^j}(z,0)\| \le SD(m+1).$$

The above inequalities give for $K = SD(m+1)$ and $z \in X$

$$\|\Gamma^{(s)}(z)\| \le K\|z_2\|^{m-s}. \tag{3.14}$$

Let ϱ' be defined for $G_{\delta,\varepsilon}$ as in Lemma 0.5. Then by Lemmas 0.1, 0.2, 0.4 and 0.5, (3.12) and since $s < m$ we can assume, taking if necessary a smaller δ, that for every function $\eta \in C^m(X,Y)$ satisfying

$$\sup\{\|\eta^{(s)}(z)\| : \ z \in X\} \leq K$$

the following inequalities

$$\sup\{\|\eta(z)\| : \ z \in X\} < \varrho'$$

and

$$\sup\{\|(G_{\delta,\varepsilon}(\cdot,\eta(\cdot)) \circ (T^{-1})_{\delta,\varepsilon})^{(s)}(z)\| : z \in X\| \leq K$$

hold. Thus we can define inductively a sequence $(\varphi_n)_{n\in\mathbb{N}_0}$ by

$$\varphi_0 = 0, \quad \varphi_{n+1} = G_{\delta,\varepsilon}(\cdot,\varphi_n(\cdot)) \circ (T^{-1})_{\delta,\varepsilon} \quad \text{for} \quad n \in \mathbb{N}_0.$$

Induction easily shows that $\sup\{\|\varphi_n^{(s)}(z)\| : \ z \in X\} \leq K$ for all positive integers n. It is also clear by Lemma 0.2 and (3.7) that all φ_n vanish with their derivatives on X_1.

Now take a $z \in X$, $\|z\| \leq \delta$. Similar argument as in proofs of Theorem 1.1 and 1.2 lead to the following inequalities for $n \in \mathbb{N}$

$$\begin{aligned}
\|\varphi_{n+1}^{(s)}(z) - \varphi_n^{(s)}(z)\| \leq & ((\|G_{\varphi_0}\| + \varepsilon)(\|T'(0)^{-1}\| + \varepsilon)^s \\
& + \delta\Delta(\delta)\sup\{\|\varphi_n^{(s)}((T^{-1})_{\delta,\varepsilon}(z)) \\
& - \varphi_{n-1}^{(s)}((T_{\delta,\varepsilon}^{-1}(z))\| : \ \|z\| \leq \delta\} \\
\leq & \ldots \leq ((\|G_{\varphi_0}\| + \varepsilon)(\|T'(0)^{-1}\| + \varepsilon)^s \\
& + \delta\Delta(\delta))^n \sup\{\|\Gamma^{(s)}((T^{-1})_{\delta,\varepsilon}^n(z))\| : \ \|z\| \leq \delta\},
\end{aligned}$$

where $\Delta : [0,+\infty) \to [0,+\infty)$ is a bounded function. Hence by (3.14) and (3.13) we get

$$\begin{aligned}
& \sup\{\|\varphi_{n+1}^{(s)}(z) - \varphi_n^{(s)}(z)\| : \ \|z\| \leq \delta\} \\
& \leq ((\|G_{\varphi_0}\| + \varepsilon)(\|T'(0)^{-1}\| + \varepsilon)^s \\
& \quad + \delta\Delta(\delta))^n (\|T'(0)|X_2)^{-1}\| + \varepsilon)^{(m-s)n} K\delta^{m-s}
\end{aligned}$$

Taking into account (3.8) we infer that for δ sufficiently small $(\varphi_n|\{z \in X : \|z\| \leq \delta\})_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the usual norm in $C^s(\{z \in X : \ \|z\| \leq \delta\}, Y)$. Hence the function ψ_2 defined by $\psi_2(z) = \lim_{n\to\infty} \varphi_n(z)$ if $\|z\| \leq \delta$, is of class C^s and is solution of (3.9) in a neighbourhood of $0 \in X$

by the definition of $(\varphi_n)_{n\in\mathbb{N}}$, $G_{\delta,\varepsilon}$ and $(T^{-1})_{\delta,\varepsilon}$. Further $\psi = \psi_1 + \psi_2$ is a local solution of (3.6) and finally $\varphi = \psi \circ R$ is a local solution of (1.1) satisfying (3.5). □

The just proved theorem is a generalization of Theorem D from [2] to the infinite dimensional case and of Theorem 3 from the same paper which concerns equation (1.50). Our theorem is also more general in the same sense than Theorem 12.2 from [6], Chapter IX, moreover, it gives an estimation of s. Observe that s increases to $+\infty$ if m does.

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