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THE BOREL FORMULA FOR INTEGRABLE DISTRIBUTIONS

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Abstract. The purpose of this paper is to give a new proof of the Borel formula for the convolution product of integrable distributions.

Let f and g be in $L^1(\mathbb{R}^n)$. Put $h(x) = \int\limits_{\mathbb{R}^n} f(x-y)g(y)dy$. The function h is called the convolution product of f and g. The function $\mathcal{F}f$, $\mathcal{F}f(\sigma) = \int\limits_{\mathbb{R}^n} e^{ix\sigma} f(x)dx$, where $x\sigma = x_1\sigma_1 + \cdots + x_n\sigma_n$ is said to be the Fourier transform of f.

THEOREM. *If f and g are in $L^1(\mathbb{R}^n)$, then the following Borel formula*

$$\mathcal{F}(f * g)(\sigma) = \mathcal{F}f(\sigma)\mathcal{F}g(\sigma) \tag{1}$$

holds.

In this note we present a natural proof of (1) when f and g are any integrable distributions. We recall now a definition of integrable distributions. Let $\mathcal{B}$ denote the set of smooth functions φ defined in $\mathbb{R}^n$ such that its all derivatives $\frac{\partial^{|\alpha|}}{\partial x^\alpha}\varphi$, $\alpha \in \mathrm{N}^n$ are bounded.

DEFINITION 1. We say the a sequence (φ_ν), $\nu \in \mathrm{N}$, $\varphi_\nu \in \mathcal{B}$ converges to the zero in the space $\mathcal{B}$ if it satisfies the following two conditions:

$$(\beta)\begin{cases} (a) & \text{there exist positive real numbers } A_\alpha \text{ such that} \\ & \left|\dfrac{\partial^{|\alpha|}}{\partial x^\alpha}\varphi_\nu(x)\right| \leqslant A_\alpha \text{ for } x \in \mathbb{R}^n \text{ and } \alpha \in \mathrm{N}^n, \\ (b) & \text{the sequence } \left(\dfrac{\partial^{|\alpha|}}{\partial x^\alpha}\varphi_\nu\right) \text{ uniformly converges to the zero} \\ & \text{on every compact set } K \in \mathbb{R}^n \text{ for } \alpha \in \mathrm{N}^n. \end{cases}$$

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DEFINITION 2. A linear continuous form Λ with respect to (β) convergence over $\mathcal{B}$ is said to be integrable distribution. The vector space of all integrable distributions will be denoted by D'_{L^1}.

We know that every integrable distribution Λ can be written as follows

$$\Lambda(\varphi) = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} \int_{\mathbb{R}^n} f_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}\varphi(x)dx \quad \text{for} \quad \varphi \in \mathcal{B}, \tag{2}$$

where $f_\alpha \in L^1$([3], p. 201).

Note that for f and g belonging to L^1 we have

$$\int_{\mathbb{R}^n} (f * g)(x)\varphi(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)\varphi(x+y)dxdy.$$

This equality we can write in the following form

$$\int_{\mathbb{R}^n} (f * g)(x)\varphi(x)dx = (f_x \otimes g_y)[\varphi(x+y)], \tag{3}$$

where $f \otimes g$ denotes the tensor product of f and g ([3], p. 106-7). Assume now that S and T are in D'_{L^1} and $\varphi \in \mathcal{B}$, then the symbol $(S_x \otimes T_y)[\varphi(x+y)]$ is sensible for $\varphi \in \mathcal{B}$. The equality (3) suggest us how to define the convolution product $S * T$ ([3], p. 204). Namely we should take

$$(S * T)(\varphi) = (S_x \otimes T_y)[(\varphi(x+y))]. \tag{4}$$

By virtue of (2) we have

(5)

$$(S * T)(\varphi) = \sum_{|\alpha|\leqslant m_1}\sum_{|\beta|\leqslant m_2} (-1)^{|\alpha+\beta|} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_\alpha(x)g_\beta(y)\frac{\partial^{|\alpha+\beta|}}{\partial x^\alpha \partial y^\beta}\varphi(x+y)dxdy.$$

Note that

$$\left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} f_\alpha \otimes \frac{\partial^{|\beta|}}{\partial y^\beta} g_\beta\right)[\varphi(x+y)]$$
$$= (-1)^{|\alpha+\beta|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_\alpha(x)g_\beta(y)\frac{\partial^{|\alpha+\beta|}}{\partial x^\alpha \partial y^\beta}\varphi(x+y)dxdy,$$

where $\frac{\partial^{|\alpha|}}{\partial x^\alpha} f_\alpha$ and $\frac{\partial^{|\beta|}}{\partial y^\beta} g_\beta$ are the distributional derivatives of order α and β of f_α and g_β respectively. For simplicity of notations we put $S_\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_\alpha$ and $T_\beta = \frac{\partial^{|\beta|}}{\partial y^\beta} g_\beta$. Hence the formula (5) can be written as follows

$$(S * T)(\varphi) = \sum_{|\alpha|\leqslant m_1}\sum_{|\beta|\leqslant m_2} (S_\alpha * T_\beta)(\varphi).$$

Taking into account the above equality we need only prove that (1) holds for S_α and T_β.

For this purpose we use the regularizations $S_\alpha * h_\varepsilon$ and $T_\beta * h_\varepsilon$, where $h_\varepsilon(x) = h_\varepsilon(x_1)\cdots h_\varepsilon(x_n)$, and $h_\varepsilon(t) = \dfrac{1}{\pi}\dfrac{\varepsilon}{\varepsilon^2+t^2}$. Exactly we have

$$(S_\alpha * h_\varepsilon)(x) = (-1)^{|\alpha|}\int_{\mathbf{R}^n} f_\alpha(x-\xi)\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}h_\varepsilon(\xi)d\xi$$

$$(T_\beta * h_\varepsilon)(y) = (-1)^{|\beta|}\int_{\mathbf{R}^n} g_\beta(y-\xi)\frac{\partial^{|\beta|}}{\partial\xi^\beta}h_\varepsilon(\xi)d\xi.$$

Since f_α, g_β and h_ε are in L^1 therefore $S_\alpha * h_\varepsilon$ and $T_\beta * h_\varepsilon$ belong to L^1, too. Moreover

$$\mathcal{F}[(S_\alpha * h_\varepsilon) * (T_\beta * h_\varepsilon)](\sigma) = \mathcal{F}f_\alpha(\sigma)\mathcal{F}g_\beta(\sigma)\mathcal{F}\left(\frac{\partial^{|\alpha|}}{\partial x^\alpha}h_\varepsilon * \frac{\partial^{|\beta|}}{\partial y^\beta}h_\varepsilon\right)(\sigma).$$

Hence

$$(6)\quad \mathcal{F}[(S_\alpha * h_\varepsilon) * (T_\beta * h_\varepsilon)](\sigma) = \mathcal{F}f_\alpha(\sigma)\mathcal{F}g_\beta(\sigma)(-i\sigma)^{|\alpha+\beta|}e^{-2\varepsilon(|\sigma_1|+\cdots+|\sigma_n|)}.$$

We shall now show that $(S_\alpha * h_\varepsilon) * (T_\beta * h_\varepsilon)$ tends to $S_\alpha * T_\beta$ in $\mathcal{S}'$ as $\varepsilon \to 0$. Indeed, note that

$$(S_\alpha * h_\varepsilon) * (T_\beta * h_\varepsilon)(x) = \frac{\partial^{|\alpha+\beta|}}{\partial x^{\alpha+\beta}}[f_\alpha * g_\beta * h_{2\varepsilon}](x).$$

Hence we obtain

$$\int_{\mathbf{R}^n}\frac{\partial^{|\alpha+\beta|}}{\partial x^{\alpha+\beta}}[f_\alpha * g_\beta * h_{2\varepsilon}](x)\varphi(x)dx = (-1)^{|\alpha+\beta|}\int_{\mathbf{R}^n}(f_\alpha * g_\beta * h_{2\varepsilon})(x)\frac{\partial^{|\alpha+\beta|}}{\partial x^{\alpha+\beta}}\varphi(x)dx$$

for $\varphi \in \mathcal{S}$ ([3], p. 233). Since $f_\alpha * g_\beta$ is in $L^1(\mathbf{R}^n)$, therefore $(f_\alpha * g_\beta) * h_{2\varepsilon}$ tends to $f_\alpha * g_\beta$ in L^1 as $\varepsilon \to 0$ ([1], p. 6).

This implies that

$$(-1)^{|\alpha+\beta|}\int\limits_{\mathbb{R}^n}[(f_\alpha * g_\beta) * h_{2\varepsilon}](x)\frac{\partial^{|\alpha+\beta|}}{\partial x^{\alpha+\beta}}\varphi(x)dx$$

tends to

$$(-1)^{|\alpha+\beta|}\int\limits_{\mathbb{R}^n}(f_\alpha * g_\beta)(x)\frac{\partial^{|\alpha+\beta|}}{\partial x^{\alpha+\beta}}\varphi(x)dx = \frac{\partial^{|\alpha+\beta|}}{\partial x^{\alpha+\beta}}(f_\alpha * g_\beta)(\varphi) =$$
$$= \left[\left(\frac{\partial^{|\alpha|}}{\partial x^\alpha}f_\alpha\right) * \left(\frac{\partial^{|\beta|}}{\partial y^\beta}g_\beta\right)\right](\varphi) = (S_\alpha * T_\beta)(\varphi).$$

By continuity of the Fourier transformation in $\mathcal{S}'$ ([3], p. 251) we infer that

$$\mathcal{F}[(S_\alpha * h_\varepsilon) * (T_\beta * h_\varepsilon)] \quad \to \quad \mathcal{F}(S_\alpha * T_\beta) \tag{7}$$

in $\mathcal{S}'$ as $\varepsilon \to 0$. Taking into account (6) by the Lebesque dominated convergence theorem one can observe that

$$\int\limits_{\mathbb{R}^n}(-i\sigma)^\alpha\mathcal{F}f(\sigma)(-i\sigma)^\beta\mathcal{F}g_\beta(\sigma)e^{-2\varepsilon(|\sigma_1|+\dots+|\sigma_n|)}\varphi(\sigma)d\sigma$$

tends to

$$\int\limits_{\mathbb{R}^n}(-i\sigma)^\alpha\mathcal{F}f(\sigma)(-i\sigma)^\beta\mathcal{F}g_\beta(\sigma)\varphi(\sigma)d\sigma = \int\limits_{\mathbb{R}^n}\mathcal{F}S_\alpha(\sigma)\mathcal{F}T_\beta(\sigma)\varphi(\sigma)d\sigma$$

as $\varepsilon \to 0$.

From this by virtue of (7) we get

$$\mathcal{F}(S_\alpha * T_\beta) = \mathcal{F}S_\alpha * \mathcal{F}T_\beta.$$

This statement finishes the proof of (1) if f and $g \in D'_{L^1}$. The formula (1) is also true if f and g are in D'_{L^2} ([2], p. 43).

References

[1] J. Barros-Neto, *An introduction to the Theory Distributions*, Marcel Dekker, Inc. New York, 1973.

[2] E.J. Beltrami and M.R. Wohlers, *Distributions and the Boundary Values of Analytic Functions*, Academic Press, New York and London, 1966.

[3] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.

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