Title: The Borel formula for integrable distributions

Author: Urszula Sztaba

THE BOREL FORMULA FOR INTEGRABLE DISTRIBUTIONS

URSZULA SZTABA

Abstract. The purpose of this paper is to give a new proof of the Borel formula for the convolution product of integrable distributions.

Let \( f \) and \( g \) be in \( L^1(\mathbb{R}^n) \). Put \( h(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy \). The function \( h \) is called the convolution product of \( f \) and \( g \). The function \( \mathcal{F}f, \mathcal{F}f(\sigma) = \int_{\mathbb{R}^n} e^{ix\sigma}f(x)dx \), where \( x\sigma = x_1\sigma_1 + \cdots + x_n\sigma_n \) is said to be the Fourier transform of \( f \).

**THEOREM.** If \( f \) and \( g \) are in \( L^1(\mathbb{R}^n) \), then the following Borel formula

\[
\mathcal{F}(f \ast g)(\sigma) = \mathcal{F}f(\sigma)\mathcal{F}g(\sigma)
\]

holds.

In this note we present a natural proof of (1) when \( f \) and \( g \) are any integrable distributions. We recall now a definition of integrable distributions. Let \( B \) denote the set of smooth functions \( \varphi \) defined in \( \mathbb{R}^n \) such that its all derivatives \( \frac{\partial^{\vert \alpha \vert}}{\partial x^\alpha}\varphi, \alpha \in \mathbb{N}^n \) are bounded.

**DEFINITION 1.** We say the a sequence \( (\varphi_\nu), \nu \in \mathbb{N}, \varphi_\nu \in B \) converges to the zero in the space \( B \) if it satisfies the following two conditions:

\[
\begin{align*}
\text{(a)} & \quad \text{there exist positive real numbers } A_\alpha \text{ such that} \\
& \quad \left| \frac{\partial^{\vert \alpha \vert}}{\partial x^\alpha}\varphi_\nu(x) \right| \leq A_\alpha \text{ for } x \in \mathbb{R}^n \text{ and } \alpha \in \mathbb{N}^n, \\
\text{(b)} & \quad \text{the sequence } \left( \frac{\partial^{\vert \alpha \vert}}{\partial x^\alpha}\varphi_\nu \right) \text{ uniformly converges to the zero} \\
& \quad \text{on every compact set } K \in \mathbb{R}^n \text{ for } \alpha \in \mathbb{N}^n.
\end{align*}
\]

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DEFINITION 2. A linear continuous form \( \Lambda \) with respect to (\( \beta \)) convergence over \( B \) is said to be integrable distribution. The vector space of all integrable distributions will be denoted by \( D'_L_1 \).

We know that every integrable distribution \( \Lambda \) can be written as follows

\[
\Lambda(\varphi) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \int_{\mathbb{R}^n} f_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha} \varphi(x) \, dx \quad \text{for} \quad \varphi \in B,
\]

where \( f_\alpha \in L^1([3], \text{p. 201}) \).

Note that for \( f \) and \( g \) belonging to \( L^1 \) we have

\[
\int_{\mathbb{R}^n} (f \ast g)(x) \varphi(x) \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) g(y) \varphi(x+y) \, dx \, dy.
\]

This equality we can write in the following form

\[
\int_{\mathbb{R}^n} (f \ast g)(x) \varphi(x) \, dx = (f \otimes g_y)[\varphi(x+y)],
\]

where \( f \otimes g \) denotes the tensor product of \( f \) and \( g \) ([3], p. 106-7). Assume now that \( S \) and \( T \) are in \( D'_L_1 \) and \( \varphi \in B \), then the symbol \( (S_x \otimes T_y)[\varphi(x+y)] \) is sensible for \( \varphi \in B \). The equality (3) suggest us how to define the convolution product \( S \ast T \) ([3], p. 204). Namely we should take

\[
(S \ast T)(\varphi) = (S \otimes T_y)[(\varphi(x+y))].
\]

By virtue of (2) we have

\[
(S \ast T)(\varphi) = \sum_{|\alpha| \leq m_1} \sum_{|\beta| \leq m_2} (-1)^{|\alpha|+|\beta|} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_\alpha(x) g_\beta(y) \frac{\partial^{|\alpha|+|\beta|}}{\partial x^\alpha \partial y^\beta} \varphi(x+y) \, dx \, dy.
\]

Note that

\[
\left( \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_\alpha \otimes \frac{\partial^{|\beta|}}{\partial y^\beta} g_\beta \right)[\varphi(x+y)]
\]

\[
= (-1)^{|\alpha|+|\beta|} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_\alpha(x) g_\beta(y) \frac{\partial^{|\alpha|+|\beta|}}{\partial x^\alpha \partial y^\beta} \varphi(x+y) \, dx \, dy,
\]

where \( \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_\alpha \) and \( \frac{\partial^{|\beta|}}{\partial y^\beta} g_\beta \) are the distributional derivatives of order \( \alpha \) and \( \beta \) of \( f_\alpha \) and \( g_\beta \) respectively. For simplicity of notations we put \( S_\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_\alpha \) and \( T_\beta = \frac{\partial^{|\beta|}}{\partial y^\beta} g_\beta \). Hence the formula (5) can be written as follows

\[
(S \ast T)(\varphi) = \sum_{|\alpha| \leq m_1} \sum_{|\beta| \leq m_2} (S_\alpha \ast T_\beta)(\varphi).
\]
Taking into account the above equality we need only prove that (1) holds for \( S_\alpha \) and \( T_\beta \).

For this purpose we use the regularizations \( S_\alpha * h_\varepsilon \) and \( T_\beta * h_\varepsilon \), where
\[
 h_\varepsilon(x) = h_\varepsilon(x_1) \cdots h_\varepsilon(x_n), \quad \text{and} \quad h_\varepsilon(t) = \frac{\varepsilon}{\pi \varepsilon^2 + t^2}.
\]
Exactly we have
\[
(S_\alpha * h_\varepsilon)(x) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f_\alpha(x - \xi) \frac{\partial^{|\alpha|}}{\partial \xi^\alpha} h_\varepsilon(\xi) d\xi
\]
\[
(T_\beta * h_\varepsilon)(y) = (-1)^{|\beta|} \int_{\mathbb{R}^n} g_\beta(y - \xi) \frac{\partial^{|\beta|}}{\partial \xi^\beta} h_\varepsilon(\xi) d\xi.
\]
Since \( f_\alpha, g_\beta \) and \( h_\varepsilon \) are in \( L^1 \) therefore \( S_\alpha * h_\varepsilon \) and \( T_\beta * h_\varepsilon \) belong to \( L^1 \), too. Moreover
\[
\mathcal{F}[(S_\alpha * h_\varepsilon) * (T_\beta * h_\varepsilon)](\sigma) = \mathcal{F}f_\alpha(\sigma) \mathcal{F}g_\beta(\sigma) \mathcal{F}\left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} h_\varepsilon * \frac{\partial^{|\beta|}}{\partial y^\beta} h_\varepsilon\right)(\sigma).
\]
Hence
\[
(6) \quad \mathcal{F}[(S_\alpha * h_\varepsilon) * (T_\beta * h_\varepsilon)](\sigma) = \mathcal{F}f_\alpha(\sigma) \mathcal{F}g_\beta(\sigma) (-i\sigma)^{|\alpha+\beta|} e^{-2\varepsilon(\sum_{i=1}^n |\sigma_i|)}.
\]
We shall now show that \( (S_\alpha * h_\varepsilon) * (T_\beta * h_\varepsilon) \) tends to \( S_\alpha * T_\beta \) in \( S' \) as \( \varepsilon \to 0 \).
Indeed, note that
\[
(S_\alpha * h_\varepsilon) * (T_\beta * h_\varepsilon)(x) = \frac{\partial^{|\alpha+\beta|}}{\partial x^\alpha \partial y^\beta} [f_\alpha * g_\beta * h_{2\varepsilon}](x).
\]
Hence we obtain
\[
\int_{\mathbb{R}^n} \frac{\partial^{|\alpha+\beta|}}{\partial x^\alpha \partial y^\beta} [f_\alpha * g_\beta * h_{2\varepsilon}](x) \varphi(x) dx = (-1)^{|\alpha+\beta|} \int_{\mathbb{R}^n} [f_\alpha * g_\beta * h_{2\varepsilon}](x) \frac{\partial^{|\alpha+\beta|}}{\partial x^\alpha \partial y^\beta} \varphi(x) dx
\]
for \( \varphi \in \mathcal{S} \) ([3], p. 233). Since \( f_\alpha * g_\beta \) is in \( L^1(\mathbb{R}^n) \), therefore \( (f_\alpha * g_\beta) * h_{2\varepsilon} \)
tends to \( f_\alpha * g_\beta \) in \( L^1 \) as \( \varepsilon \to 0 \) ([1], p. 6).
This implies that
\[ (-1)^{|\alpha+\beta|} \int_{\mathbb{R}^n} [(f_\alpha * g_\beta) * h_\varepsilon](x) \frac{\partial^{|\alpha+\beta|}}{\partial x^\alpha \beta} \varphi(x) dx \]
tends to
\[ (-1)^{|\alpha+\beta|} \int_{\mathbb{R}^n} (f_\alpha * g_\beta)(x) \frac{\partial^{|\alpha+\beta|}}{\partial x^\alpha \beta} \varphi(x) dx = \frac{\partial^{|\alpha+\beta|}}{\partial x^\alpha \beta} (f_\alpha * g_\beta)(\varphi) = \]
\[ = \left[ \left( \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_\alpha \right) * \left( \frac{\partial^{|\beta|}}{\partial y^\beta} g_\beta \right) \right] (\varphi) = (S_\alpha * T_\beta)(\varphi). \]

By continuity of the Fourier transformation in $S'$ ([3], p. 251) we infer that
\[ \mathcal{F}[(S_\alpha * h_\varepsilon) * (T_\beta * h_\varepsilon)] \rightarrow \mathcal{F}(S_\alpha * T_\beta) \]
in $S'$ as $\varepsilon \rightarrow 0$. Taking into account (6) by the Lebesque dominated convergence theorem one can observe that
\[ \int_{\mathbb{R}^n} (-i\sigma)^\alpha \mathcal{F}f(\sigma)(-i\sigma)^\beta \mathcal{F}g_\beta(\sigma) e^{-2\varepsilon(1+\ldots+1)} \varphi(\sigma) d\sigma \]
tends to
\[ \int_{\mathbb{R}^n} (-i\sigma)^\alpha \mathcal{F}f(\sigma)(-i\sigma)^\beta \mathcal{F}g_\beta(\sigma) \varphi(\sigma) d\sigma = \int_{\mathbb{R}^n} \mathcal{F}S_\alpha(\sigma) \mathcal{F}T_\beta(\sigma) \varphi(\sigma) d\sigma \]
as $\varepsilon \rightarrow 0$.

From this by virtue of (7) we get
\[ \mathcal{F}(S_\alpha * T_\beta) = \mathcal{F}S_\alpha * \mathcal{F}T_\beta. \]

This statement finishes the proof of (1) if $f$ and $g \in D'_{L^1}$. The formula (1) is also true if $f$ and $g$ are in $D'_{L^2}$ ([2], p. 43).
REFERENCES


INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
40-007 KATOWICE
POLAND