You have downloaded a document from
RE-BUŚ
repository of the University of Silesia in Katowice

Title: Multiplicities of Peano maps : on a less known theorem by Hurewicz

Author: Wojciech Dębski, Jerzy Mioduszewski

Citation style: Dębski Wojciech, Mioduszewski Jerzy. (1995). Multiplicities of Peano maps : on a less known theorem by Hurewicz. "Annales Mathematicae Silesianae" (Nr 9 (1995), s. 11-15).

Uznanie autorstwa - Użycie niekomercyjne - Bez utworów zależnych Polska - Licencja ta zezwala na rozpowszechnianie, przedstawianie i wykonywanie utworu jedynie w celach niekomercyjnych oraz pod warunkiem zachowania go w oryginalnej postaci (nie tworzenia utworów zależnych).

UNIWERSYTET ŚLĄSKI W KATOWICACH
Biblioteka Uniwersytetu Śląskiego
Ministerstwo Nauki i Szkolnictwa Wyższego

Annales Mathematicae Silesianae 9. Katowice 1995, 11–15

Prace Naukowe Uniwersytetu Śląskiego nr 1523

MULTIPLICITIES OF PEANO MAPS: ON A LESS KNOWN THEOREM BY HUREWICZ

WOJCIECH DĘBSKI AND JERZY MIODUSZEWSKI

Abstract. It is shown that the values of two highest multiplicities of Peano maps cannot be the values of openess.

This adds a new detail to the results by Hurewicz (1933), concerning dimension raising maps, when they are restricted to maps from the closed interval into the plane.

A *Peano map* is a continuous map of a closed interval of reals onto a subset of the plane which has a non-empty interior. The number of points at which a given value is assumed is called the *multiplicity* of this value. A *finite-to-one* map is a map the multiplicities all values of which are finite.

Hurewicz (1933) proved that a finite-to-one map having only two multiplicities for its values cannot be Peano.

Theorem by Hurewicz belongs to rather less known and cannnot be found in handbooks, even on this level of generality as was just expressed. In fact, the original statement of this theorem is much more general, and elementary tools which will be used here in the proof of its special case are not sufficient to prove it.

The original statement and comments concerning the tools needed for proof will be given in the end part of the article.

1. An auxiliary theorem. A value y of a map $f : X \to Y$ between topological spaces will be called a *value of openess* if for each point x at which it is assumed it lies in the interior of the image of any neighbourhood U of x, i.e. if $y \in \operatorname{int} f(U)$, in symbols.

Values which are not values of openess are for continuous maps – under natural restriction concerning the spaces – rather exceptional: in the case of

AMS (1991) subject classification: 54F15.

real functions the values of openess are the values the levels of which do not cross the graphs at local extrema.

Let X be a compact metric space and let $\mathcal{B}$ be its given countable base. Let f be a continuous map from X onto a metric space Y. Delete from Y the boundaries of sets $f(A)$, where A are the closures of sets from $\mathcal{B}$. In result, countably many closed nowhere dense sets (boundaries of closed sets are nowhere dense; the sets $f(A)$ are closed as they are compact) are deleted. Thus, a first category F_σ–set is delete. *The remainder is a dense G_δ-set consisting only of values of openess of f.*

The conclusion of the above reasoning is the content of the theorem which will be used in the latter proof.

2. The statement of the theorem. We shall prove the following somewhat more detailed form of the announced theorem.

THEOREM. *The values with the highest multiplicities cannot be the values of openess of a Peano map, if they lie in the interior of the image.*

Since the values of openess of a Peano map form a dense subset of the image (this follows from the auxiliary theorem from the preceding section), from the stated above theorem will follow that a Peano map must have at least three values with different multiplicities. This is the content of the theorem announced at the beginning.

3. Proof of the theorem. Let $f : I \to E^2$ be a continuous finite-to-one map from closed interval of reals into the plane and let y be the value of f which lies in the interior of the image. Let

$$f^{-1}(y) = \{x_1, \dots, x_k\}, \qquad \text{where} \quad x_i \neq x_j \;\text{ if }\; i \neq j.$$

Assume that y is a value of openess and that the number k is one of the two highest multiplicities of f.

Let U_j be mutually disjoint intervals around x_j. The value y lies in the interior of each of sets $f(U_j)$. Let W be a domain around y contained in the intersection of all sets int $f(U_j)$.

Fix one of the sets U_j, let U_1 be that set, and take an open interval U with $x_1 \in U$ contained in U_1 and so small that $f(\overline{U})$ is contained in the domain W. We still have $y \in \operatorname{int} f(\overline{U})$ (we have even $y \in \operatorname{int} f(U)$, as y is a value of openess). So, we infer that the boundary of $f(\overline{U})$ disconnects the domain W.

According to known theorems on disconnection of the plane, a closed set disconnecting a plane domain must contain a continuum which not reduces to a point (*). Let C be a not reduced to a point continuum contained in the boundary of $f(\overline{U})$ and such that the values of f assumed at ends of U do not belong to C.

Let $c \in C$. We have $c \in f(U)$, since the values at ends of U do not belong to C. Since the boundary of the set $f(U)$ is nowhere dense in the domain W, the point c is a limit point of points from W lying outside of $f(\overline{U})$. These points are values of f assumed at points of U_1 (as $W \subset f(U_1)$) lying outside of $\overline{U}$. Their limit points lie outside of U and they are mapped under f into c. Thus, the point c is the value of f at at least two points of the interval U_1, one of which lies in U and the other outside of U. Since the point c – as a point lying in W – is assumed as a value of f on each of the (mutually disjoint) intervals U_j, the multiplicity of c as a value of f is at least $k+1$.

In the case when k is the highest multiplicity of f, the proof is finished in view of the obvious contradiction.

In the case which remains, consider the map f restricted to $f^{-1}(C)$. From the preceding reasoning it follows that the values of this restricted map have all the same multiplicity: the highest multiplicity for values of f. Denote by r this multiplicity.

Let c' be a value of openess of the map f restricted to $f^{-1}(C)$. We have

$$f^{-1}(c') = \{c'_1, \dots, c'_r\}, \qquad \text{where,} \quad c'_i \neq c'_j \ \text{ if } \ i \neq j.$$

Let V_j be mutually disjoint open neighbourhoods of c'_j in $f^{-1}(C)$. The value c' lies in the interiors (with respect to C) of sets $f(V_j)$.

Let C' be a not reducing to a point continuum contained in the intersection of interiors with respect to C of sets $f(V_j)$ (**). We have

$$f^{-1}(C') = C'_1 \cup \ldots \cup C'_r,$$

where $C'_j = V_j \cap f^{-1}(C')$. The sets $C'{}_j$ are mutually disjoint (as V_j are) and are compact (this follows from the compactness of $f^{-1}(C')$ and disjointness of open sets V_j). Since C' lies in the intersection of the sets $f(U_j)$, each point of C' is assumed as a value on each set V_j exactly once in view of disjointness of V_j. Thus, we infer that f, when restricted to C'_j, is (for each j) an one-to-one map onto C', thus a homeomorphism onto C', in view of continuity and compactness.

It follows that all C'_j are not reducing to points, subcontinua of the interval I, thus are mutually disjoint closed, not reducing to points, intervals of I. In consequence, the continuum C' is an arc lying in the plane interior of the set $f(I)$.

Let c'' be a point of the arc C' being not an end of C'. Points c''_j of the counterimage

$$f^{-1}(c') = \{c''_1, \dots, c''_r\}$$

are interior points in intervals C'_j. Thus, the union $C'_1 \cup \ldots \cup C'_r$ is a neighbourhood of $f^{-1}(c'')$; the last set is the counterimage of the full map f, since the number r is the highest multiplicity of values of f. The image of a neighbourhood of the full counterimage of a value is a neighbourhood of that value. Applying this rule to the neighbourhood $C'_1 \cup \ldots \cup C'_r$ of $f^{-1}(c')$, we get a contradiction, since the image of that neighbourhood lies on the arc C' which is nowhere dense in $f(I)$.

4. Comments. A Peano map, if is finite-to one, must have (as was proved) at least three multiplicities for its values. *Is it true that all the couples of three or more numbers realize as the sets of multiplicities of Peano maps?* The set of multiplicities of the classical map constructed by Peano is 1, 2 and 4, and Polya constructed a Peano map with the set of multiplicities 1, 2 and 3.

These maps, as well as Peano maps constructed by Hilbert and Sierpiński, are *irreducible*, i.e. such that the image diminishes when a non-empty open set is deleted from the domain. The values of irreducible map with the multiplicity 1 form then a dense G_δ-set in the image; this set coincides with the set of values of openess of the map. *Must – as in all the known irreducible examples – values with the multiplicity* 2 *appear ?*

Does there exist an irreducible Peano map with the multiplicities 1 and ∞ *only ?*

The original theorem by Hurewicz is much more general and asserts roughly speaking that *if a metric separable space differs not so much from manifolds* – satisfying the Hurewicz condition (α) – *then a continuous map from this space onto a metric space the dimension of which exceeds by* m *or more the dimension of the domain, has at least* $m + 2$ *multiplicities for its values.*

In the case considered here, the maps raised the dimension by 1 or more, thus the number 3 in the conclusion.

The proof of the full theorem by Hurewicz needs some tools from advanced parts of the theory of dimension, e.g. a theorem of Freudenthal estimating the dimension of sets of values which multiplicities are higher than a given one. The authors of the article think that each escape into generality from the generality considered here needs these advanced tools.

The best source of classical Peano maps is the book by Sierpiński (1947); p. 117.

(*) *A compact zero–dimensional set*, thus a compact set which does not contain reducing to points continua, *does not disconnect the plane.* More generally: a compact set of dimension $\leq n-2$ does not disconnect E^n.

(**) In the proof of the existence of such a continuum the Janiszewski

Lemma should be used: *components of an open subset of a continuum accumulate at the boundary of that set.*

References

[1] W. Hurewicz, *Über dimensionserhöhende stetige Abbildungen*, Journal für reine und angewandte Mathematik **169** (1933).
[2] W. Sierpiński, *Introduction to the theory of sets and topology* (in Polish), Warszawa 1947 (second edition).

Uniwersytet Śląski
Instytut Matematyki
ul. Bankowa 14
40–007 Katowice