Title: Some remarks on the Daróczy equation

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SOME REMARKS ON THE DARÓCZY EQUATION

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Abstract. The general solution of the functional equation

\[ f(x) = f(x + 1) + f(x(x + 1)), \]

considered both on \((0, +\infty)\) and \(\mathbb{R}\), are studied. Constructions of odd and even solutions are given.

In this paper we deal with the functional equation

\[ f(x) = f(x + 1) + f(x(x + 1)) \]

and its real solution, generally defined on \((0, +\infty)\). Some problems concerning this equation was posed by Z.Daróczy during the XXIV ISFE in South Hadley [3]. The main problem was solved by M.Laczkovich and R.Redheffer [5]; see also [6], [1], [2], [4]. In part 1 we investigate the general solution \(f : (0, +\infty) \to \mathbb{R}\) of (1) in the spirit of [6] by Z.Moszner. Next we give another construction of the general solution of the Daróczy equation which bases on an equivalence relation on \((0, +\infty)\). In part 3 we present constructions of real solutions of equation (1) defined on \(\mathbb{R}\). In particular, we construct of all the odd and all the even solutions of (1). Finally, in part 4 we introduce another equation, a generalization of (1), and give some informations on its solutions under the assumption that there exists the limit \(\lim_{x \to +\infty} xf(x)\), like it is in papers of K. Baron [1], [2] and W. Jarczyk [4].

1. Let us start with a simple remark: putting \(x\) instead of \(x(x + 1)\) in (1) we obtain

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REMARK 1. A function \( f: (0, +\infty) \to \mathbb{R} \) is a solution of (1) if and only if
\[
(2) \quad f(x) = f \left( \frac{\sqrt{1+4x} - 1}{2} \right) - f \left( \frac{\sqrt{1+4x} + 1}{2} \right)
\]
for \( x \in (0, +\infty) \).

The following theorem brings a description of the general solution of (1). In a special case \( a = 6 \) it reduces to the result of Z. Moszner [6].

**Theorem 1.** If \( a \in (2, 6] \) then for every real function \( f_0 \) defined on \( \left[ \frac{\sqrt{1+4a} - 1}{2}, a \right) \) there exists exactly one solution \( f: (0, +\infty) \to \mathbb{R} \) of (1) which is an extension of \( f_0 \).

**Proof.** Define \( \varphi: [0, +\infty) \to \mathbb{R} \) by
\[
(3) \quad \varphi(x) := \frac{\sqrt{1+4x} - 1}{2},
\]
observe that
\[
0 < \varphi(x) < x \quad \text{for} \quad x \in (0, +\infty), \quad \varphi(0) = 0,
\]
\[
\varphi^{-1}(x-1) > x \quad \text{for} \quad x \in (2, +\infty)
\]
and let \((a_n : n \in \mathbb{Z}), (b_n : n \in \mathbb{N})\) be the sequences such that
\[
a_0 = \varphi(a) \quad \text{and} \quad \varphi(a_n) = a_{n-1} \quad \text{for} \quad n \in \mathbb{Z},
\]
\[
b_0 = a \quad \text{and} \quad b_n = \varphi^{-1}(b_{n-1} - 1) \quad \text{for} \quad n \in \mathbb{N}.
\]
The sequence \((b_n : n \in \mathbb{N})\) is strictly increasing to infinity. Hence we can find the number \( N \in \mathbb{N} \) such that
\[
b_{N-1} < a + 1 \quad \text{and} \quad b_N \geq a + 1.
\]
Then
\[
a_1 = a < b_N = \varphi^{-1}(b_{N-1} - 1) < \varphi^{-1}(a) = \varphi^{-1}(a_1) = a_2.
\]
Define now functions \( f_{1,1}, f_{1,2}, \ldots, f_{1,N+1} \) in the following way:
\[
f_{1,1}(x) := f_0(\varphi(x)) - f_0(\varphi(x) + 1), \quad x \in [a_1, b_1),
\]
\[
f_{1,n}(x) := f_0(\varphi(x)) - f_{1,n-1}(\varphi(x) + 1), \quad x \in [b_{n-1}, b_n), n = 2, \ldots, N,
\]
\[
f_{1,N+1}(x) := f_0(\varphi(x)) - f_{1,N}(\varphi(x) + 1), \quad x \in [b_N, a_2),
\]
and put

\[ f_1 := \bigcup_{j=1}^{N+1} f_{1,j}. \]

Also the sequence \((a_n : n \in \mathbb{Z})\) is strictly increasing and \(\lim_{n \to -\infty} a_n = 0\), \(\lim_{n \to +\infty} a_n = +\infty\). For every positive integer \(n \geq 2\) define the function \(f_n : [a_n, a_{n+1}) \to \mathbb{R}\) by putting

\[
    f_n(x) := \begin{cases} 
        f_{n,1}(x), & x \in [a_n, \varphi^{-1}(a_n - 1)), \\
        f_{n,2}(x), & x \in [\varphi^{-1}(a_n - 1), a_{n+1}), 
    \end{cases}
\]

where

\[
    f_{n,1}(x) := f_{n-1}(\varphi(x)) - f_{n-1}(\varphi(x) + 1), \quad x \in [a_n, \varphi^{-1}(a_n - 1)),
\]

\[
    f_{n,2}(x) := f_{n-1}(\varphi(x)) - f_{1,n}(\varphi(x) + 1), \quad x \in [\varphi^{-1}(a_n - 1), a_{n+1}).
\]

To define \(f_n : [a_n, a_{n+1}) \to \mathbb{R}\) for negative integers we put

\[
    f_{-1}(x) := f_0(x + 1) + f_0(x(x + 1)) \quad \text{for} \quad x \in [a_{-1}, a_0),
\]

\[
    f_{n-1}(x) := \begin{cases} 
        f_0(x + 1) + f_n(x(x + 1)), & x \in [a_{n-1}, a_n) \cap [a_0 - 1, a_1 - 1), \\
        f_{-1}(x + 1) + f_n(x(x + 1)), & x \in [a_{n-1}, a_n) \cap [a_{-1} - 1, a_0 - 1), 
    \end{cases}
\]

for \(n \leq -1\). Finally we define \(f : (0, +\infty) \to \mathbb{R}\) by

\[
    f(x) := f_n(x) \quad \text{for} \quad x \in [a_n, a_{n+1}), \ n \in \mathbb{Z}.
\]

It follows from the definition of \(f_n\) for \(n \geq 1\) that (2) holds for \(x \geq a_1\), whereas the definition of \(f_n\) for \(n \leq -1\) gives (1) for positive \(x < a_0\). Hence, since \(x \leq a_1\) implies \(\varphi(x) < a_0\), we have

\[
    f(\varphi(x)) = f(\varphi(x) + 1) + f(x) \quad \text{for} \quad x \in (a_0, a_1).
\]

In other words, \(f\) is a solution of (2). According to Remark 1 it is also a solution of (1).

Finally, if \(\tilde{f} : (0, +\infty) \to \mathbb{R}\) is a solution of (1) and an extension of \(f_0\) then \(f_n(x) = \tilde{f}(x)\) for \(x \in [a_n, a_{n+1})\) and \(n \in \mathbb{Z}\) whence \(f = \tilde{f}\).

\[\square\]

**Corollary 1.** If two solutions of (1) defined on \((0, +\infty)\) coincides on \([\sqrt{1+4a-1}/2, a)\) for some \(a \in (2, 6]\), then they are identical.
Later (in Remark 2 below) we shall show that the above theorem doesn't hold for $a = 2$. However, we have the following result.

**Theorem 2.** Let $f_1, f_2 : (0, +\infty) \to \mathbb{R}$ are solutions of (1) such that either

(i) there exist the limits

$$\lim_{x \to 2^+} f_1(x), \quad \lim_{x \to 2^+} f_2(x),$$

and at least one of them is finite;

or

(ii) there exists an $\varepsilon > 0$ such that

$$f_1(x) \geq f_2(x) \quad \text{for} \quad x \in (2, 2 + \varepsilon).$$

If

$$f_1 \mid_{[1, 2]} = f_2 \mid_{[1, 2]}$$

then

$$f_1 = f_2.$$

**Proof.** Defining

$$f := f_1 - f_2$$

we observe that $f$ is a solution of (1) vanishing on $[1, 2)$. We shall show that it vanishes on $[1, 6)$. Putting $x = 1$ in (1) we obtain $f(2) = 0$. Fix $x_0 \in (2, 6)$, define $\varphi : (0, +\infty) \to \mathbb{R}$ by (3) and the sequence $(x_n : n \in \mathbb{N})$ putting

$$x_n := \varphi(x_{n-1}) + 1.$$

We can easy show that this sequence is strictly decreasing to 2. In particular,

$$\varphi(x_n) \in \varphi((2, 6)) = (1, 2).$$

Hence

$$0 = f(\varphi(x_n)) = f(\varphi(x_n) + 1) + f(\varphi(x_n)(\varphi(x_n) + 1)) = f(x_{n+1}) + f(x_n)$$
i.e.

$$f(x_{n+1}) = -f(x_n) \quad \text{for} \quad n \in \mathbb{N}_0.$$

This gives

$$f(x_n) = (-1)^n f(x_0) \quad \text{for} \quad n \in \mathbb{N}.$$
In case (i) the sequence \( (f(x_n) : n \in \mathbb{N}) \) has a limit whence \( f(x_0) = 0 \). In case (ii) we have \( f(x_n) \geq 0 \) for \( n \) large enough and so \( f(x_0) = 0 \) as well. Thus we have proved that \( f \) vanishes on \((1,6)\) and it follows from Corollary 1 that \( f \) vanishes everywhere. It means that \( f_1 = f_2 \).

Now we shall explain more precisely non-uniqueness in extending functions from \([1, 2)\) to solutions of Daróczy equation on \((0, +\infty)\).

**Remark 2.** For any solution \( f_1 : (0, +\infty) \to \mathbb{R} \) of (1), for any \( a \in (2, 6) \) and for any function \( u : [\frac{\sqrt{1+4a+1}}{2}, a) \to \mathbb{R} \) there exists a solution \( f_2 : (0, +\infty) \to \mathbb{R} \) of (1) such that

\[
    f_1(x) = f_2(x) \quad \text{for} \quad x \in (0, 2]
\]

and

\[
    f_1(x) - f_2(x) = u(x) \quad \text{for} \quad x \in \left[\frac{\sqrt{1+4a+1}}{2}, a\right).
\]

We precede our proof of this remark by the following lemma.

**Lemma 1.** If a solution of (1) on \((0, +\infty)\) vanishes on \((1, 2)\) then it vanishes on \((0, 2]\).

**Proof.** Let \( f : (0, +\infty) \to \mathbb{R} \) be a solution of (1) vanishing on \((1, 2]\). Define \( \varphi : (0, +\infty) \to \mathbb{R} \) by (3) and the sequence \( (x_n : n \in \mathbb{N}) \) putting

\[
    x_0 := 2 \quad \text{and} \quad x_n := \varphi(x_{n-1}) \quad \text{for} \quad n \in \mathbb{N}.
\]

This sequence is strictly decreasing to zero and \( x_1 = 1 \). Moreover, if \( n \in \mathbb{N} \) and \( x \in (x_{n+1}, x_n) \) then \( x + 1 \in (x_1, x_0] \) and \( x(x + 1) \in (x_n, x_{n-1}] \). Hence \( f \) vanishes on \((x_1, x_0]\) and if \( f \) vanishes on \((x_n, x_{n-1}]\) then, as a solution of (1), it vanishes also on \((x_{n+1}, x_n]\) \( \square \).

**Proof of Remark 2.** We have to define a solution \( f : (0, +\infty) \to \mathbb{R} \) of (1) which vanishes on \((0, 2]\) and coincides with \( u \) on \([\frac{\sqrt{1+4a+1}}{2}, a)\). Define \( \psi : (2, +\infty) \to \mathbb{R} \) by

\[
    \psi(x) := \frac{\sqrt{1+4x+1}}{2}
\]

and the sequence \( (c_n : n \in \mathbb{N}) \) putting

\[
    c_1 := a \quad \text{and} \quad c_{n+1} := \psi(c_n) \quad \text{for} \quad n \in \mathbb{N}.
\]

\( \ast \)
This sequence is strictly decreasing to 2. Hence for every \( n \in \mathbb{N} \) we can define the function \( f_n : [c_{n+1}, c_n) \to \mathbb{R} \) by

\[
f_n(x) := (-1)^{n-1}u(\psi^{-(n-1)}(x)) \quad \text{for} \quad x \in [c_{n+1}, c_n).
\]

Putting

\[
f_0(x) := \begin{cases} 
  f_n(x), & x \in [c_{n+1}, c_n), \quad : : n \in \mathbb{N}, \\
  0, & x \in \left[\frac{\sqrt{1+4a}-1}{2}, 2\right],
\end{cases}
\]

and using Theorem 1 we obtain a solution \( f : (0, +\infty) \to \mathbb{R} \) of (1) which is an extension of \( f_0 \); in particular \( f \) coincides with \( u \) on \( \left[\frac{\sqrt{1+4a}-1}{2}, a\right] \). Now we show that \( f \) vanishes on \( (0, 2] \). On virtue of Lemma 1 and the definition of \( f_0 \) it is enough to check that \( f \) vanishes on \( (1, \sqrt{\frac{1+4a-1}{2}}) \). Let \( x \in (1, \sqrt{\frac{1+4a-1}{2}}) \). Then \( x + 1 \in (2, c_2) \) and there exists an \( n \geq 2 \) such that \( x + 1 \in [c_{n+1}, c_n) \). Hence

\[
x(x + 1) = \psi^{-1}(x + 1) \in [\psi^{-1}(c_{n+1}), \psi^{-1}(c_n)) = [c_n, c_{n-1})
\]

and

\[
f(x) = f(x + 1) + f(x(x + 1)) \\
= (-1)^{n-1}u(\psi^{-(n-1)}(x + 1)) + (-1)^{n-2}u(\psi^{-(n-2)}(x(x + 1))) \\
= (-1)^{n-1}u(\psi^{-(n-1)}(x + 1)) + (-1)^{n-2}u(\psi^{-(n-2)}(\psi^{-(1)}(x + 1))) \\
= (-1)^{n-1}u(\psi^{-(n-1)}(x + 1)) + (-1)^{n-1}u(\psi^{-(n-1)}(x + 1)) = 0.
\]

\[\Box\]

2. In this section we present another construction of solutions of Daróczy equation and we give two examples of discontinuous at each point solutions: such that there exists the limit at infinity and such that this limit does not exist.

**THEOREM 3.** There exists a partition \( \mathcal{X} \) of \( (0, +\infty) \) consisting of countable and dense subsets of \( (0, +\infty) \) such that

\[
(4) \quad \text{if} \; \; X \in \mathcal{X} \; \; \text{and} \; \; x \in X \; \; \text{then} \; \; x + 1, \; x(x + 1) \in X;
\]

in particular, a function \( f : (0, +\infty) \to \mathbb{R} \) is a solution of (1) iff for every \( X \in \mathcal{X} \) the function \( f |_X \) does.

**PROOF.** Define \( \varphi : (0, +\infty) \to \mathbb{R} \) by (3) and \( \tau : (0, +\infty) \to \mathbb{R} \) by

\[
\tau(x) = x + 1,
\]

and
put
\[ \Phi = \{ \varphi, \varphi^{-1}, \tau, \tau^{-1} \} \]
and define the relation \( \sim \) on \((0, +\infty)\) by
\[ x \sim y \iff y = \varphi_1(\ldots(\varphi_n(x)\ldots) \text{ for some } \varphi_1, \ldots, \varphi_n \in \Phi. \]

One can easily check that it is an equivalence relation and thus defines a partition \( \mathcal{X} \) of \((0, +\infty)\) consisting of its equivalence classes. It is clear that if \( X \in \mathcal{X} \) then \( X \) is countable and (4) holds. We shall show that \( X \) is also dense in \((0, +\infty)\). Suppose for the contrary that there exist \( a, b \in (0, +\infty) \) such that \( a < b \) and \( (a, b) \cap X = \emptyset \). Then
\[ \emptyset = \varphi^{-1}((a, b)) \cap \varphi^{-1}(X) = (\varphi^{-1}(a), \varphi^{-1}(b)) \cap X \]
and so (by induction)
\[ (\varphi^{-n}(a), \varphi^{-n}(b)) \cap X = \emptyset \quad \text{for every } n \in \mathbb{N}. \]
Since \( \varphi^{-1}(x) > x \) for \( x \in (0, +\infty) \) and \( (\varphi^{-1})'(x) \geq 2a + 1 \) for \( x \geq a \) we have
\[ \varphi^{-(n+1)}(b) - \varphi^{-(n+1)}(a) \geq (2a + 1)(\varphi^{-n}(b) - \varphi^{-n}(a)) \]
for every \( n \in \mathbb{N} \), whence
\[ \lim_{n \to +\infty} (\varphi^{-n}(b) - \varphi^{-n}(a)) = +\infty. \]
Consequently there exists an \( n \in \mathbb{N} \) such that
\[ \varphi^{-n}(b) - \varphi^{-n}(a) > 1. \]
Let \( x \in X \) and fix an integer \( k \) such that
\[ x + k \in (\varphi^{-n}(a), \varphi^{-n}(b)). \]
Then
\[ \tau^k(x) = x + k \in (\varphi^{-n}(a), \varphi^{-n}(b)) \cap X, \]
a contradiction. \( \Box \)

Theorem 3 allows us to give some interesting examples.

**Remark 3.** (i) There exists a solution \( f : (0, +\infty) \to (0, +\infty) \) of (1) which is discontinuous at each point and such that the limit
\[ \lim_{x \to +\infty} f(x). \]
does not exist.

(ii) There exist a solution \( f : (0, +\infty) \to (0, +\infty) \) of (1) which is discontinuous at each point and such that

\[
\lim_{x \to +\infty} f(x) = 0.
\]

**Proof.** Let \( \mathcal{X} \) be a partition of \( (0, +\infty) \) with the properties mentioned in Theorem 3, fix a non-constant function \( c : \mathcal{X} \to (0, +\infty) \) and define a solution \( f : (0, +\infty) \to (0, +\infty) \) of (1) by

\[
f(x) := \frac{c(X)}{x} \quad \text{for } x \in X, \ X \in \mathcal{X}.
\]

It is clear that \( f \) is discontinuous at each point. If \( c \) is bounded then (6) holds and we have (ii). Assume \( c \) is unbounded. We shall prove that limit (5) does not exists. For, let \( (X_n : n \in \mathbb{N}) \) be a sequence of elements of \( \mathcal{X} \) with \( \lim_{n \to +\infty} c(X_n) = +\infty \) and for every \( n \in \mathbb{N} \) choose an \( x_n \in (c(X_n), 2c(X_n)) \cap X_n \). Then \( \lim_{n \to +\infty} x_n = +\infty \) and

\[
f(x_n) > \frac{1}{2} \quad \text{for } n \in \mathbb{N}.
\]

If the limit (5) existed we would have

\[
\lim_{n \to +\infty} f(x_n) = \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} f |_{X} (x) = 0
\]

for every \( X \in \mathcal{X} \), a contradiction with (7). \( \square \)

3. In this part of the paper we shall show a construction of all the solutions of (1) defined on \( \mathbb{R} \). Let us start with two simple lemmas.

**Lemma 2.** If \( g : \mathbb{R} \to \mathbb{R} \) is a solution of (1) then the function \( G : \mathbb{R} \to \mathbb{R} \) defined by

\[
G(x) := g(x) + g(-x)
\]

is periodic with period 1.

**Proof.** Fix \( x \in \mathbb{R} \). Then, according to (1),

\[
g(-x - 1) = g(-x) + g(x(x + 1)) = g(-x) + [g(x) - g(x + 1)]
\]
LEMMA 3. Every solution $g: (-1, +\infty) \to \mathbb{R}$ of (1) has a unique extension to a solution $f: \mathbb{R} \to \mathbb{R}$ of (1).

PROOF. Define $G: [0, 1) \to \mathbb{R}$ by (8) and $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} g(x), & x \in (-1, +\infty), \\ G\{x\} - g(-x), & x \in (-\infty, -1], \end{cases}$$

where $\{x\}$ denotes the fractal part of $x$. Observe that for every $x \in (0, 1)$ we have

$$G\{-x\} = G(1 - x) = g(1 - x) + g(x - 1) = [g(-x) - g(-x(-x + 1))] + g(x - 1) = g(-x) - g(x(x - 1)) + g(x - 1) = g(x) + g(-x) = G\{x\}$$

whence

$$G\{-x\} = G\{x\} \quad \text{for} \quad x \in \mathbb{R}.$$  

Now we shall show that $f$ is a solution of (1). Of course (1) holds for $x \in (-1, +\infty)$. Assume now that $n \in \mathbb{N}$ and (1) holds for every $x \in (-n, +\infty)$. Then for $x \in (-n - 1, -n]$ we have

$$f(x) = G\{-x\} - f(-x) = G\{-x - 1\} - f(-x) = f(x + 1) + g(-x - 1) - f(-x) = f(x + 1) + g(-x - 1) - f(-x) = f(x + 1) + f(x(x + 1))$$

and so $f$ is a solution of (1). Finally, if $\tilde{f}$ is an extension $\tilde{g}$ to a solution of (1) then applying Lemma 2 we see that

$$\tilde{f}(x) + g(-x) = \tilde{f}(\{x\}) + g(-\{x\}) = g(\{x\}) + g(-\{x\}) = G(\{x\})$$

for $x \in \mathbb{R}$, whence for $x \in (-\infty, -1]$ we obtain

$$f(x) = G(\{x\}) - g(-x) = \tilde{f}(x) + \tilde{f}(-x) - \tilde{f}(-x) = \tilde{f}(x)$$

which ends the proof. \qed
THEOREM 4. If $a \in (2,6]$ then for every real function $f_0$ defined on the set
\[
\left[-\frac{1}{2}, -\frac{1}{4}\right) \cup \{0\} \cup \left[\frac{\sqrt{1+4a-1}}{2}, 2\right) \cup (2,a)
\]
there exists exactly one solution $f : \mathbb{R} \to \mathbb{R}$ of (1) which is an extension of $f_0$.

PROOF. First of all let us observe that any solution of (1) defined on $[0, +\infty)$ vanishes at 1 and 2. Hence, extending $f_0$ onto $[\frac{\sqrt{1+4a-1}}{2}, a]$ by putting $f_0(2) = 0$ and applying Theorem 1 we see that $f_0$ has a unique extension to a solution $\tilde{f}_0 : (0, +\infty) \to \mathbb{R}$ of (1). Extend now $\tilde{f}_0$ onto $[0, +\infty)$ by putting $\tilde{f}_0(0) = f_0(0)$. Then $\tilde{f}_0$ is the unique extension of $f_0$ to a solution of (1) defined on $[0, +\infty)$. Define $\varphi : [-\frac{1}{4}, 0) \to [-\frac{1}{2}, 0)$ by (3) and the sequence $(x_n : n \in \mathbb{N}_0)$ putting
\[
x_0 := -\frac{1}{2} \quad \text{and} \quad x_n := \varphi^{-1}(x_{n-1}) \quad \text{for} \quad n \in \mathbb{N}.
\]
This sequence strictly increases to zero. For every positive integer $n$ define a function $f_n : [x_n, x_{n+1}) \to \mathbb{R}$ by
\[
f_n(x) := f_{n-1}(\varphi(x)) - \tilde{f}_0(\varphi(x) + 1), \quad x \in [x_n, x_{n+1}).
\]
The formula
\[
\tilde{f}_1 := f_n(x) \quad \text{for} \quad x \in [x_n, x_{n+1}) \quad \text{and} \quad n \in \mathbb{N}_0
\]
defines a function $\tilde{f}_1 : [-\frac{1}{2}, 0) \to \mathbb{R}$. With the aid of $\tilde{f}_0$ and $\tilde{f}_1$ define $\tilde{f}_2 : (-1, -\frac{1}{2}) \to \mathbb{R}$ putting
\[
\tilde{f}_2(x) := \tilde{f}_0(x + 1) + \tilde{f}_1(x(x + 1)).
\]
Finally we define $\tilde{f} : (-1, +\infty) \to \mathbb{R}$ by
\[
\tilde{f} := \tilde{f}_0 \cup \tilde{f}_1 \cup \tilde{f}_2.
\]
It is easy to see that $\tilde{f}$ is the unique extension of $f_0$ to a solution of (1) defined on $(-1, +\infty)$. An application of Lemma 3 ends the proof. □

The following simple theorem describes even solution of (1).

THEOREM 5. The only even solution of (1) on $\mathbb{R}$ is the zero function.
PROOF. If $f: \mathbb{R} \to \mathbb{R}$ is an even solution of (1) then an application of Lemma 2 shows that $f$ is periodic with period 1 and (1) gives

$$f(x(x + 1)) = 0 \quad \text{for} \quad x \in \mathbb{R}.$$ 

In particular, $f(x) = 0$ for $x \in [0, +\infty)$ and, as $f$ is even, $f = 0$. \hfill \Box

All the odd solutions of equation (1) defined on $\mathbb{R}$ describes the following theorem.

**Theorem 6.** If $a \in (2, 6]$ then for every real function $f_0$ defined on the set

$$\left(0, \frac{1}{2}\right) \cup \left[\frac{\sqrt{1 + 4a + 1}}{2}, a\right)$$

there exists exactly one odd solution $f: \mathbb{R} \to \mathbb{R}$ of (1) which is an extension of $f_0$.

**Proof.** It is easy to observe that the function $\tilde{f}_0: (0, 1) \to \mathbb{R}$ given by

$$\tilde{f}_0(x) := \begin{cases} 
    f_0(x), & x \in \left(0, \frac{1}{2}\right), \\
    \frac{1}{2}f_0\left(\frac{1}{4}\right), & x = \frac{1}{2}, \\
    f_0(x(1-x)) - f_0(1-x), & x \in \left(\frac{1}{2}, 1\right),
\end{cases}$$

satisfies

$$\tilde{f}_0(x) + \tilde{f}_0(1-x) = \tilde{f}_0(x(1-x)) \quad \text{for} \quad x \in (0, 1).$$

Define $\psi: (1, +\infty) \to \mathbb{R}$ by $\psi(x) = (x - 1)x$ and $(x_n : n \in \mathbb{N}_0)$ by

$$x_0 := 1 \quad \text{and} \quad x_{n+1} := \psi^{-1}(x_n) \quad \text{for} \quad n \in \mathbb{N}.$$ 

This is a strictly increasing sequence with the limit equal to 2. For every non-negative integer $n$ define a function $g_n:[x_n, x_{n+1}) \to \mathbb{R}$ putting

$$g_n(x) := 0 \quad \text{and} \quad g_0(x) := \tilde{f}_0(x - 1) - \tilde{f}_0(\psi(x)), \quad x \in (x_0, x_1),$$

$$g_n(x) := \tilde{f}_0(x - 1) - g_{n-1}(\psi(x)), \quad x \in [x_n, x_{n+1}), \quad n \in \mathbb{N},$$
and a function $\tilde{f}_1: [1, 2) \to \mathbb{R}$ as

$$\tilde{f}_1 := g_0 \cup g_1 \cup g_2 \cup \ldots.$$ 

Consider also a sequence $(a_n : n \in \mathbb{N}_0)$ such that

$$a_0 := a \quad \text{and} \quad a_{n+1} := \psi^{-1}(a_n) \quad \text{for} \quad n \in \mathbb{N}.$$ 

This sequence strictly decreases to 2. For every positive integer $n$ define a function $h_n: [a_n, a_{n-1}) \to \mathbb{R}$ putting

$$h_1(x) := f_0(x), \quad x \in [a_1, a_0),$$

$$h_n(x) := \tilde{f}_1(x - 1) - h_{n-1}(\psi(x)), \quad x \in [a_n, a_{n-1}), \quad n \geq 2,$$

and a function $\tilde{f}_2: (2, a) \to \mathbb{R}$ as

$$\tilde{f}_2 := h_1 \cup h_2 \cup \ldots.$$ 

Furthermore, let

$$f_1 := \tilde{f}_0 \cup \tilde{f}_1 \cup \tilde{f}_2$$

and extend $f_1$ onto $[0, a)$ assuming additionally

$$f_1(0) := 0, \quad f_1(2) := 0.$$ 

It follows from (11)–(14) that

$$f_1(x) = f_1(x - 1) - f_1(\psi(x)) \quad \text{for} \quad x \in (1, a),$$

i.e. $f_1$ satisfy (1) for $x \in (0, a - 1)$. Applying Theorem 1 to the function $f_1$ restricted to $[\frac{\sqrt{1+4a^{-1}}}{2}, a)$ we obtain exactly one solution $f_2: (0, +\infty) \to \mathbb{R}$ of (1) which coincides with $f_1$ on $[\frac{\sqrt{1+4a^{-1}}}{2}, a)$. As the function

$$f_1 \mid_{(0,a)} \cup f_2 \mid_{[a, +\infty)}$$

coincides with $f_1$ on $[\frac{\sqrt{1+4a^{-1}}}{2}, a)$ and is a solution of (1) it follows that (from Corollary 1) that the function (15) equals $f_2$. In particular $f_2$ is an extension of $f_1$. Consequently $f_2$ is an extension of $f_0$, $f_2(1) = 0$ and (cf. (10))

$$f_2(x) + f_2(1 - x) = f_2(x(1 - x)) \quad \text{for} \quad x \in (0, 1).$$
Let \( f: \mathbb{R} \to \mathbb{R} \) be the odd extension of \( f_2 \). We shall check that \( f \) is a solution of (1). Of course (1) holds for \( x \in [0, +\infty) \). If \( x \in (-\infty, -1) \), then

\[
f(x + 1) + f(x(x + 1)) = -f_2(-x - 1) + f_2(x(x + 1))
\]

\[
= -f_2(-x - 1) + f_2((-x - 1)(-x - 1) + 1))
\]

\[
= -f_2(-x - 1) + f_2(-x - 1) - f_2(-x) = f(x).
\]

Next, if \( x \in (-1, 0) \) then using (16) we have

\[
f(x + 1) + f(x(x + 1)) = f_2(x + 1) - f_2(-x(x + 1)) = f_2(x).
\]

Finally, since \( f(-1) = -f(1) = 0 \) we see that (1) holds for \( x = -1 \) as well.

To end the proof, assume that \( \tilde{f}: \mathbb{R} \to \mathbb{R} \) is an odd solution of (1) and an extension of \( f_0 \). It follows from (9) that \( \tilde{f} |_{(0, 1)} = \tilde{f}_0 \) whereas (1) gives \( \tilde{f}(1) = 0 \). Hence and from (11) and (12) it follows that \( \tilde{f} |_{[1, 2]} = \tilde{f}_1 \). This jointly with (13) shows that \( \tilde{f} |_{(2, a)} = \tilde{f}_2 \). Since (1) gives \( \tilde{f}(2) = 0 \) we have \( \tilde{f} |_{(0, a)} = \tilde{f}_1 \). Applying now Theorem 1 we obtain \( \tilde{f} |_{(0, +\infty)} = f_2 \) and \( \tilde{f} = f \).

\[\square\]

4. Fix a positive real number \( a \). Of course,

\[
\frac{1}{x} = \frac{a}{x(x + a)} + \frac{1}{x + a} \quad \text{for} \quad x \in (0, +\infty).
\]

In the other words, the function \( f: (0, +\infty) \to \mathbb{R} \) defined by \( f(x) := 1/x \) is a solution of

\[
f(x) = f(x + a) + f\left(\frac{x(x + a)}{a}\right)
\]

as well of

\[
f(x) = f(x + a) + af(x(x + a)).
\]

In the case where \( a = 1 \) each of these two equations reduce to (1). In fact (17) is equivalent to (1) for every \( a > 0 \). For, if \( f: (0, +\infty) \to \mathbb{R} \) is a solution of (1) then

\[
f\left(\frac{x}{a}\right) = f\left(\frac{x}{a} + 1\right) + f\left(\frac{x}{a}\left(\frac{x}{a} + 1\right)\right) \quad \text{for} \quad x \in (0, +\infty)
\]

and putting \( \tilde{f}(x) := f(x/a) \) we obtain

\[
\tilde{f}(x) = \tilde{f}(x + a) + \tilde{f}\left(\frac{x(x + a)}{a}\right) \quad \text{for} \quad x \in (0, +\infty),
\]
i.e. \( \tilde{f} \) is a solution of (17). However, as it follows from Theorem 8 below, in general equations (18) and (1) are not equivalent.

In this part of the paper we shall examine solutions of (18) under the assumption that there exists the limit \( \lim_{x \to +\infty} xf(x) \) (see Baron [1], [2]) and we obtain solutions of (18) which are not of the form \( \frac{c}{x} \) on the whole interval \((0, +\infty)\).

**Theorem 7.** Let \( a \in (0, +\infty) \). If \( f: (0, +\infty) \to \mathbb{R} \) is a solution of (18) such that there exists the limit

\[
\lim_{x \to +\infty} xf(x),
\]

then this limit is finite and

\[
f(x) = \frac{c}{x} \quad \text{for} \quad x \in (0, +\infty) \cap [1 - a, +\infty)
\]

with \( c \) being the limit (19).

Similarly as K. Baron did in [2], let us start with the following lemma.

**Lemma 4.** Let \( a \in (0, +\infty) \) and \( f: (0, +\infty) \to \mathbb{R} \) be a solution of (18). If there exists an \( M > 0 \) such that for some \( c \in \mathbb{R} \) we have

\[
f(x) < -c \quad \text{for} \quad x > M
\]

then

\[
f(x) \leq \frac{c}{x} \quad \text{for} \quad x \in (0, +\infty) \cap [1 - a, +\infty).
\]

**Proof.** Replacing \( f \) by \( \tilde{f}(x) = f(x) - c/x, x > 0 \), we may assume that \( c = 0 \). Fix arbitrarily \( x_0 \in (0, +\infty) \cap (1 - a, +\infty) \) and define the sequence \( (x_n : n \in \mathbb{N}) \) by

\[
x_{n+1} := \min\{x_n + a, x_n(x_n + a)\} \quad \text{for} \quad n \in \mathbb{N}
\]

It is easy to see that the sequence \( (x_n : n \in \mathbb{N}) \) increases to infinity. Using induction and (18) one can see that for every positive integer \( n \) there exists a sequence

\[(l_1, \ldots, l_{2^n})\]

of non-negative integers and a sequence

\[(\alpha_1, \ldots, \alpha_{2^n})\]
of numbers not smaller than $x_n$ such that

$$f(x_0) = \sum_{i=1}^{2^n} a_i f(\alpha_i).$$

(20)

Now, if $n$ is a positive integer such that $x_n > M$ then (20) gives $f(x_0) \leq 0$. This proves that $f$ is nonpositive on $(0, +\infty) \cap (1 - a, +\infty)$. If $1 - a > 0$ then applying (18) we obtain that also $f(1 - a) \leq 0$. \hfill \Box

**Proof of Theorem 7.** When having Lemma 4, our Theorem 7 may be proved as the main result of [2]. For the sake of completeness we repeat this proof here.

Assume the limit (19) equals $-\infty$ and fix arbitrarily a real number $c$. Then there exists an $M > 0$ such that

$$xf(x) \leq c$$

for $x > M$.

Hence and from the lemma we obtain

$$xf(x) \leq c$$

for $x \in (0, +\infty) \cap (1 - a, +\infty)$,

which leads to a contradiction as $c$ was fixed arbitrarily. The case when the limit (19) equals $+\infty$ reduces to the previous one by considering the function $-f$. Up to now we have proved that the limit (19) is finite. Denote it by $c$ and fix arbitrarily an $\varepsilon > 0$. Then there exists an $M > 0$ such that

$$xf(x) \leq c + \varepsilon$$

for $x > M$.

Hence and from the lemma we obtain

$$xf(x) \leq c + \varepsilon$$

for $x \in (0, +\infty) \cap (1 - a, +\infty)$.

Consequently, as the positive number $\varepsilon$ has been fixed arbitrarily we have

$$xf(x) \leq c$$

for $x \in (0, +\infty) \cap (1 - a, +\infty)$.

Applying it to the function $-f$ we shall obtain the reverse inequality which ends the proof. \hfill \Box

**Theorem 8.** If $a \in (0, 1)$, $x_0 \in [1 - 2a, 1 - a) \cap (0, 1)$ and $x_1 := \frac{\sqrt{a^2 + 4x_0} - a}{2}$ then for every $c \in \mathbb{R}$ and for every $u : [x_0, x_1) \rightarrow \mathbb{R}$ there exists exactly one solution $f : (0, +\infty) \rightarrow \mathbb{R}$ of (18) which is an extension of $u$ and

$$\lim_{x \to +\infty} xf(x) = c;$$

(21)
moreover, \( f \) is continuous iff \( u \) is continuous and

\[
(22) \quad \lim_{x \to x_1} u(x) = au(x_0) + \frac{c}{x_1 + a}.
\]

**Proof.** As in the proof of Lemma 4 we may assume that \( c = 0 \). Define \( \varphi : (0, +\infty) \to \mathbb{R} \) by

\[
\varphi(x) := \frac{\sqrt{a^2 + 4x} - a}{2}.
\]

Putting \( \varphi(x) \) instead of \( x \) in (18) we obtain that \( f : (0, +\infty) \to \mathbb{R} \) is a solution of (18) if and only if it is a solution of

\[
(23) \quad f(x) = a^{-1} f(\varphi(x)) - a^{-1} f(\varphi(x) + a).
\]

Let \( (x_n : n \in \mathbb{Z}) \) be the sequence such that

\[
x_{n+1} = \varphi(x_n) \quad \text{for} \quad n \in \mathbb{Z}.
\]

Of course it is strictly increasing and \( \lim_{n \to -\infty} x_n = 0, \lim_{n \to +\infty} x_n = 1 - a \). Given \( u : [x_0, x_1) \to \mathbb{R} \) define a function \( f_0 : [x_0, +\infty) \to \mathbb{R} \) by

\[
f_0(x) := \begin{cases} 
a^n u(\varphi^{-n}(x)), & x \in [x_n, x_{n+1}), n \in \mathbb{N}_0, \\
0, & x \in [1 - a, +\infty).
\end{cases}
\]

Clearly, \( f_0 \) is an extension of \( u \). We shall prove that \( f_0 \) is a solution of (23). It is obvious that (23) holds for \( x \in [1 - a, +\infty) \). Let \( x \in [x_0, 1 - a) \). Then there exists an \( n \in \mathbb{N}_0 \) such that \( x \in [x_n, x_{n+1}) \) and

\[
\varphi(x) \in \varphi([x_n, x_{n+1})) = [x_{n+1}, x_{n+2}).
\]

Since \( x \geq x_0 \geq 1 - 2a \), we have \( \varphi(x) + a \geq 1 - a \) and \( f_0(\varphi(x) + a) = 0 \). Consequently,

\[
a^{-1} f_0(\varphi(x)) - a^{-1} f_0(\varphi(x) + a) = a^{-1} f_0(\varphi(x))
= a^{-1} a^n u(\varphi^{-(n+1)}(\varphi(x)))
= a^{-n} u(\varphi^{-n}(x)) = f_0(x).
\]

Furthermore, if \( f_0 \) is continuous then so is \( u \) and (22) holds. Assume now \( u \) is continuous and (22) holds. It is easy to see that then \( f_0 \mid_{[x_0, 1 - a]} \) is continuous and \( u \) is bounded, say | \( u(x) \) | \( \leq M \) for \( x \in [x_0, x_1) \), whence | \( f_0(x) \) | \( \leq a^n M \) for \( x \in [x_n, x_{n+1}], n \in \mathbb{N} \) and, consequently, \( \lim_{x \to 1-a} f(x) = 0 \). This proves
that the function \( f_0 \) is continuous iff \( u \) is continuous and (22) holds. Now define \( f_n : [x_n, +\infty) \to \mathbb{R} \) for negative integers \( n \) by

\[
 f_n(x) := \begin{cases} 
 f_{n+1}(x), & x \in [x_{n+1}, +\infty), \\
 a^{-1}f_{n+1}(\varphi(x)) - a^{-1}f_{n+1}(\varphi(x) + a), & x \in [x_n, x_{n+1}),
\end{cases}
\]

and observe that if for some negative integer \( n \) the function \( f_{n+1} \) is a continuous solution of (23) then \( f_n \) does. Hence we can define a function \( f : (0, +\infty) \to \mathbb{R} \) by

\[
 f := f_0 \cup f_{-1} \cup f_{-2} \cup \ldots.
\]

This function is a solution of (23), and so of (18), an extension of \( u \), and \( f \) is continuous iff \( f_0 \) does. Moreover, (21) holds as \( f \) vanishes on \([1 - a, +\infty)\). Finally, if \( \tilde{f} \) is an extension of \( u \) to a solution of (18) such that

\[
 \lim_{x \to +\infty} xf(x) = 0
\]

then applying Theorem 7 and an induction we see that \( \tilde{f} \) coincides with \( f_n \) on \([x_n, +\infty)\) for non-positive integers \( n \) whence \( \tilde{f} = f \).

\( \square \)

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