Title: Regular solutions of some functional equations in the indeterminate case

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REGULAR SOLUTIONS OF SOME FUNCTIONAL EQUATIONS
IN THE INDETERMINATE CASE

Abstract. The paper deals with the existence and uniqueness of regular solutions of the equation \( \varphi(x) = h(x, \varphi(f(x))) \). Also in the indeterminate case the existence of solutions of \( \varphi(f(x)) = g(x, \varphi(x)) \) is studied.

In the present paper we shall consider the following functional equations

\begin{align*}
(1) & \quad \varphi(x) = h(x, \varphi(f(x))), \\
(2) & \quad \varphi(f(x)) = g(x, \varphi(x)),
\end{align*}

where \( \varphi : I = [0, a) \to R, \ 0 < a \leq \infty \) is an unknown function.

The phrase \textit{regular solution} used in the title will have the following meaning: “a solution which is continuous in the whole interval I and possesses a right-side derivative at the point zero”.

The problem of regular solutions of linear functional equations is contained in [1] and [2]. The theory of continuous solutions of equations (1) and (2) has been developed in [4], [5], [6], [7], [8].

§ 1. Let \( I \) be an interval \( [0, a) \), \( 0 < a \leq \infty \) and let \( \Omega \) be a neighbourhood of \( (0, 0) \in R^2 \). Assume that the given functions \( f, g \) and \( h \) fulfil the following conditions.

(i) The function \( f : I \to R \) is continuous, strictly increasing, there exists \( f(0^+) \neq 0 \) and \( 0 < f(x) < x \) in \( I \setminus \{0\} \).

(ii) The function \( h : \Omega \to R \) is continuous, there exist \( c > 0, \ d > 0 \) and a continuous function \( \gamma : [0, c) \subset I \to R \) such that

\begin{equation}
|h(x, y_1) - h(x, y_2)| \leq \gamma(x) |y_1 - y_2| \text{ in } U \cap \Omega,
\end{equation}

where \( U : 0 \leq x < c, \ |y| < d \). Moreover, there exist \( A \) and \( B \) such that

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\( h(x, y) = Ax + By + R(x, y) \),

where

\( R(x, y) = o\left(\sqrt{x^2 + y^2}\right), \ (x, y) \to (0, 0). \)

For every \( x \in I \) we denote

\( \Omega_x := \{ y : (x, y) \in \Omega \} \)

and

\( A_x := \{ h(x, y) : y \in \Omega_x \} \).

(iii) For every \( x \in I \), \( \Omega_x \) is an open interval and \( A_{f(x)} = \Omega_x \).

(iv) The function \( g : \Omega \to R \) is continuous for every \( x \in I \). For every fixed \( x \in I \) the function \( g \) as a function of \( y \) is invertible. There exist \( c > 0, d > 0 \) and a continuous function \( \gamma : [0, c) \subset I \to R \) such that

\[ |g(x, y_1) - g(x, y_2)| \leq \gamma(x)|y_1 - y_2| \text{ in } U \cap \Omega, \]

where \( U : 0 \leq x < c, |y| < d \) and \( \gamma \) has a positive bound in \([0, c)\). Moreover, there exist \( A \) and \( B \neq 0 \) such that

\( g(x, y) = Ax + By + R(x, y), \)

where

\( R(x, y) = o\left(\sqrt{x^2 + y^2}\right), \ (x, y) \to (0, 0). \)

For every \( x \in I \) we denote

\( V_x := \{ g(x, y) : y \in \Omega_x \}, \)

where \( \Omega_x \) is given by (6).

(v) For every \( x \in I \), \( \Omega_x \) is an open interval and \( V_x = \Omega_{f(x)} \).

The indeterminate case

\( f'(0) \gamma(0) = 1 \)

for equation (1) and

\( \frac{f'(0)}{\gamma(0)} = 1 \)

for equation (2) will be considered in this paper.

\section{2} Let us consider equation (1) and let \( \varphi(x) = x \psi(x) \). Then equation (1) is of the form

\( \psi(x) = H(x, \varphi[f(x)]), \)

where

\( H(x, z) := \begin{cases} \frac{1}{x} h(x, f(x) z), & \text{for } x \neq 0 \\ A + B f'(0) z, & \text{for } x = 0 \end{cases} \)
is defined in the set
\[ \Omega^1 = \{(x, z) : (x, f(x)z) \in \Omega\}. \]

For an arbitrary \( x \in I \) we denote
\[ \Omega^1_x = \{z : (x, z) \in \Omega^1\} \]
and
\[ \Delta^1_x = \{H(x, z) : z \in \Omega^1_x\} \]

**Lemma 1.** Assume hypotheses (i), (ii), (iii) and condition (9). Then the function \( H \) given by the formula (12) fulfills the following conditions:
(a) \( H \) is continuous in a neighbourhood of the point \((0, \eta)\), where \( \eta \) is arbitrary constant.
(b) For every \( \eta \), there exist a \( \delta \in (0, c] \) and a \( d_1 > 0 \) such that
\[ |H(x, z_1) - H(x, z_2)| < \gamma_1(x) |z_1 - z_2| \quad \text{for} \quad x \in [0, \delta), \quad |z - \eta| < d_1, \]
where
\[ \gamma_1(x) = \begin{cases} \frac{f(x)}{x} \gamma(x), & \text{for} \quad x \in (0, \delta) \\ 1, & \text{for} \quad x = 0 \end{cases} \]
is a continuous function.
(c) For every \( x \in I \), \( \Omega^1_x \) is an open interval and \( \Delta^1_{f(x)} \subseteq \Omega^1_x \).

**Proof.** Ad (a). It suffices to show, that the function \( H \) is continuous at the point \((0, \eta)\). We have
\[ \lim_{x \to 0^+} H(x, z) = \lim_{z \to \eta} \frac{1}{x} h[x, f(x) z] = \lim_{z \to \eta} \left[A + B \frac{f(x)}{x} z + \frac{1}{x} R(x, f(x) z)\right]. \]
From (5) we obtain
\[ \lim_{x \to 0^+} \frac{R(x, f(x) z)}{\sqrt{x^2 + |f(x) z|^2}} = 0 \]
whereas (i) implies
\[ \left| \frac{R(x, f(x) z)}{\sqrt{x^2 + |f(x) z|^2}} \right| \leq \left| \frac{R(x, f(x) z)}{x \sqrt{1 + z^2}} \right|. \]
Therefore,
\[ \lim_{x \to 0^+} H(x, z) = A + B f'(0) \eta = A + B \eta, \]
which finishes our proof in case (a).

Ad (b). The function \( f \) is continuous and \( f(0) = 0 \); thus there exist a \( \delta \in (0, c] \) and a \( d_1 > 0 \) such that for every \( x \in (0, \delta) \) and \( |z - \eta| < d_1 \) the inequality \(|f(x)z| < d|z - \eta| \) holds. Therefore, the inequality
\[ |H(x, z_1) - H(x, z_2)| = \left| \frac{1}{x} [h(x, f(x) z_1) - h(x, f(x) z_2)] \right| \leq \]
\[ \leq \frac{f(x)}{x} \gamma(x) |z_1 - z_2| \quad \text{for} \quad x \neq 0 \]
holds and
\[ |H(0, z_1) - H(0, z_2)| = |B| f'(0) |z_1 - z_2|. \]
From (3) we have
\[ |B| = |h_y(0, 0)| \leq \gamma(0). \]
Consequently,
\[ |B| f'(0) \leq \gamma(0) f'(0) = 1. \]
Condition (13) is fulfilled with the function \( \gamma_1 \) given by formula (14). We have also
\[ \lim_{x \to 0} \frac{f(x)}{x} \gamma(x) = f'(0) \gamma(0) = 1. \]
This implies that \( \gamma_1 \) is continuous in \([0, c)\).

Ad (c). We may assume that \( \Omega \) is of the form
\[ \Omega : \begin{cases} 0 \leq x < a \\ \alpha_1(x) < y < \alpha_2(x), \end{cases} \]
where \( \alpha_1(x) < 0 \) and \( \alpha_2(x) > 0 \) for \( x \in [0, a) \). It is easily seen that
\[ \Omega_1 : \begin{cases} 0 \leq x < a \\ \frac{\alpha_1(x)}{f(x)} < z < \frac{\alpha_2(x)}{f(x)} \text{ for } x \neq 0 \\ z \text{ is arbitrary for } x = 0. \]
In particular, we have \( \Omega_0^1 = (-\infty, +\infty) \) whence \( \Omega_{f_1}^1 \subseteq \Omega_0^1 \). For \( x \neq 0 \) we have
\[ \Omega_{f(x)}^1 = \{ v : v = H[f(x), z], z \in \Omega_{f(x)}^1 \} = \{ v : v = \frac{1}{f(x)} h[f(x), f^2(x) z], z \in \left( \frac{\alpha_1[f(x)]}{f^2(x)}, \frac{\alpha_2[f(x)]}{f^2(x)} \right) \} = \{ v : f(x) v = h[f(x), y], y \in (\alpha_1[f(x)], \alpha_2[f(x)]) = \Omega_{f(x)} \}. \]
From (iii) we obtain
\[ \Omega_{f(x)} = \{ h[f(x), y] : y \in \Omega_{f(x)} \} \subseteq \Omega_x \]
whence
\[ \frac{\alpha_1(x)}{f(x)} < v < \frac{\alpha_2(x)}{f(x)}. \]
Therefore
\[ \Omega_{f(x)}^1 \subseteq h \left( \frac{\alpha_1(x)}{f(x)}, \frac{\alpha_2(x)}{f(x)} \right) = \Omega_{f(x)}^1. \]
The proof of Lemma 1 is complete.
From Lemma 1 and condition (11) we obtain.

**LEMMA 2.** We assume (i), (ii), (iii). If \( \psi \) is a continuous solution of equation (11) in \( I \), then \( \varphi(x) = x \psi(x) \) is a regular solution of equation (1)
in I and such that \( \varphi(0) = 0 \). If \( \varphi \) is a regular solution of equation (1) in I and such that \( \varphi(0) = 0 \) and \( \varphi'(0) = \eta \), then the function

\[
\psi(x) = \begin{cases} 
\frac{\varphi(x)}{x} & \text{for } x \neq 0 \\
\varphi'(0) & \text{for } x = 0
\end{cases}
\]

yields a continuous solution of equation (11) in I such that \( \psi(0) = \eta \).

The uniqueness of regular solutions depends essentially on the behaviour of the sequence

\[
(15) \quad \Gamma_n(x) := \left[ \gamma_1[f^n(x)] \sum_{i=0}^{n-1} \gamma_i \right],
\]

where \( \gamma_1 \) is defined by (14).

THEOREM 1. We assume (i), (ii) (iii) and (9). Let \( \eta \) be an arbitrary constant. If there exist an \( M > 0 \) and a \( \delta_1 \in (0, \delta] \) (where \( \delta \) is the constant from Lemma 1) such that

\[
\Gamma_n(x) \leq M, \ n = 0, 1, \ldots \text{ for } x \in [0, \delta_1),
\]

then equation (1) has at most one regular solution in I fulfilling conditions \( \varphi(0) = 0 \), \( \varphi'(0) = \eta \).

Proof. From our hypotheses and from Lemma 1 it follows that the assumptions of Theorem 1 from [4] are fulfilled. Thus equation (11) has at most one continuous solution in I fulfilling the condition \( \psi(0) = \eta \). Now, our assertion results from Lemma 2.

REMARK 1. If equation (11) has a continuous solution, then

\[
(16) \quad H(0, \eta) = \eta.
\]

The definition of function \( H \) implies that equation (16) assumes the form

\[
A + B f'(0) \eta = \eta.
\]

Let

\[
(17) \quad H(x) := |H(x, \eta) - \eta| = \begin{cases} 
\frac{1}{x} h(x, f(x), \eta) - \eta & \text{for } x \neq 0 \\
0 & \text{for } x = 0.
\end{cases}
\]

THEOREM 2. We assume (i), (ii), (iii) and condition (9). If (for a fixed \( \eta \) fulfilling equation (16)) there exists a \( \delta_2 > 0 \) such that

\[
(18) \quad \sum_{n=0}^{\infty} \Gamma_n(x) \left| f^n(x) \right|
\]

is uniformly convergent in \([0, \delta_2)\) then equation (1) has a regular solution in I fulfilling conditions \( \varphi(0) = 0 \), \( \varphi'(0) = \eta \).

Proof. From the hypotheses of our theorem and from Lemma 1 it follows that the assumptions of Theorem 3 from [4] concerning equa-
tion (11) are satisfied. Consequently, equation (11) possesses a continuous solution in $I$ fulfilling the condition $\varphi(0) = \eta$. Now, our assertion results from Lemma 2.

§ 3. If we put $\varphi(x) = x \psi(x)$ in equation (2), then we come to

\[
\psi[f(x)] = K[x, \psi(x)],
\]

where the function

\[
K(x, z) = \begin{cases} 
\frac{1}{f(x)} \frac{g(x, xz)}{f'(x)} & \text{for } x \neq 0 \\
\frac{A}{f(0)} + \frac{B}{f'(0)} & \text{for } x = 0
\end{cases}
\]

is defined in the region

\[
\Omega^2 = \{ (x, z) : (x, xz) \in \Omega \}.
\]

For any fixed $x \in I$ we put

\[
\Omega^1_x = \{ y : (x, y) \in \Omega^2 \}
\]

and

\[
V^2_x = \{ K(x, z) : z \in \Omega^2 \}.
\]

**Lemma 3.** Suppose that assumptions (i), (iv), (v) and condition (10) are satisfied. Then the function $K(x, z)$ fulfils the following conditions:

(a) $K(x, z)$ is continuous in a neighbourhood of each point $(0, \eta)$, where $\eta$ is an arbitrary constant.

(b) For every real $\eta$ there exist constants $\delta \in (0, c]$ and $d_1 > 0$ such that

\[
|K(x, z_1) - K(x, z_2)| \leq \gamma_1(x)|z_1 - z_2| \text{ for } 0 \leq x < \delta, \ |z - \eta| < d_1,
\]

where

\[
\gamma_1(x) = \begin{cases} 
\frac{x}{f(x)} \gamma(x) & \text{for } x \in (0, \delta) \\
1 & \text{for } x = 0
\end{cases}
\]

is a continuous function.

(c) For every $x \in I$ the set $\Omega^2_x$ is an open interval and $V^2_x = \Omega^2_{f(x)}$.

(d) For any fixed $x \in I$ the function $K(x, z)$ is invertible with respect to $z$.

**Proof.** Ad (a). In order to prove that $K(x, z)$ is continuous it suffices to show its continuity at a point $(0, \eta)$;

\[
\lim_{{x \to 0^+ \atop z \to \eta}} K(x, z) = \lim_{{x \to 0^+ \atop z \to \eta}} \frac{1}{f(x)} g(x, xz) = \lim_{{x \to 0^+ \atop z \to \eta}} \left[ A \frac{x}{f(x)} + B \frac{x}{f(x)} z + \frac{1}{f(x)} R(x, xz) \right].
\]

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Hypothesis (8) implies
\[ \lim_{x \to 0^+} \frac{R(x, xz)}{x \sqrt{1 + z^2}} = 0, \]
whence
\[ \lim_{x \to 0^+} \frac{1}{f(x)} R(x, xz) = \lim_{x \to 0^+} \frac{R(x, xz)}{x} = 0. \]

Consequently
\[ \lim_{x \to 0^+} K(x, z) = \frac{A}{f'(0)} + \frac{B}{f'(0)} = K(0, \eta), \]
which finishes the proof of (a).

Ad (b). Evidently, for every real \( \eta \) one can find a \( \delta \in (0, c] \) and a positive \( d_1 \) such that the inequality \( |xz| < d \) is satisfied whenever \( x \in (0, \delta) \) and \( |z - \eta| < d_1 \). Applying condition (7) we get
\[ |K(x, z_1) - K(x, z_2)| = \left| \frac{1}{f(x)} [g(x, xz_1) - g(x, xz_2)] \right| \leqslant \frac{B}{f'(0)} |x| |z_1 - z_2| \text{ for } x \neq 0. \]

For \( x = 0 \) we have
\[ |K(0, z_1) - K(0, z_2)| = \frac{|B|}{f'(0)} |z_1 - z_2|. \]
Condition (7) implies also that
\[ |B| = |g_y(0, 0)| \leqslant \gamma(0). \]

Since \( 0 < f(x) < x \), we have \( f'(0) = \lim_{x \to 0} \frac{f(x)}{x} > 0 \), because \( f'(0) \neq 0 \) by assumption. Consequently, on account of (10) we have
\[ \frac{|B|}{f'(0)} \leqslant \frac{\gamma(0)}{f'(0)} = 1. \]
This proves that condition (21) is satisfied with \( \gamma_1(x) \) defined by (22).
Since
\[ \lim_{x \to 0^+} \frac{x}{f(x)} \gamma(x) = \frac{\gamma(0)}{f'(0)} = 1, \]
the function \( \gamma_1(x) \) is continuous in \([0, \delta)\) which completes the proof of condition (b).

Ad (c). We may assume that the domain \( \Omega \) has the form
\[ \Omega : \begin{cases} 0 \leqslant x < a \\ a_1(x) < y < a_2(x), \end{cases} \]
where \( a_1(x) < 0 \) and \( a_2(x) > 0 \). It is easy to check that, in such a case, the region \( \Omega^2 \) is of the form

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\[ \begin{align*}
\Omega^2 : & \left\{ \begin{array}{l}
0 \leq x < a \\
\frac{a_1(x)}{x} < z \leq \frac{a_2(x)}{x} \quad \text{for } x \neq 0, \\
\text{z is arbitrary} \quad \text{for } x = 0.
\end{array} \right. \\
\end{align*} \]

Since \( f(0) = 0 \) we have \( \Omega^2_{f(0)} = \Omega^2_0 = (-\infty, \infty) \). On the other hand

\[ K(0, z) = \frac{A}{f(0)} + \frac{B}{f(0)} z, \text{ whence } V^2_0 = (-\infty, \infty) \]

i.e. \( V^2_0 = \Omega^2_{f(0)} \). For \( x \neq 0 \) we have

\[ V^2_x = \left\{ v : v = \frac{1}{f(x)} g(x, xz), z \in \left( \frac{a_1(x)}{x}, \frac{a_2(x)}{x} \right) \right\} = \{ v : f(x) v = g(x, y), y \in \Omega_x \}. \]

Assumption (v) implies

\[ \{ f(x) v : f(x) v = g(x, y), y \in \Omega_x \} = (a_4[f(x)], a_2[f(x)]) \]

whence

\[ V^2_x = \left( \frac{a_1[f(x)]}{f(x)}, \frac{a_2[f(x)]}{f(x)} \right) = \Omega^2_{f(x)}, \]

which proves our assertion (c).

Ad (d). Since \( K(0, z) = \frac{A}{f(0)} + \frac{B}{f(0)} z \) and \( B \neq 0 \) by assumption, function \( K(0, z) \) is invertible. For \( x \neq 0 \) the function \( K(x, z) \) is a one-to-one mapping with respect to \( z \) because \( g(x, y) \) is invertible as a function of the second variable (with an arbitrarily fixed \( x \in I \)). This proves (d).

The following lemma is a simple consequence of equation (19) and Lemma 3.

**LEMMA 4.** Assume (i), (iv) and (v). If \( \psi \) is a continuous solution of equation (19) then the function \( \varphi(x) = x \psi(x) \) is a regular solution of equation (2) fulfilling the condition \( \varphi(0) = 0 \). If \( \varphi \) is a regular solution of equation (2) in \( I \) such that \( \varphi(0) = 0 \) and \( \varphi'(0) = \eta \), then the function

\[ \psi(x) = \begin{cases} 
\frac{\varphi(x)}{x} & \text{for } x \neq 0 \\
\varphi'(0) & \text{for } x = 0
\end{cases} \]

is a continuous solution of (19) fulfilling the condition \( \psi(0) = \eta \).

Let \( \Gamma_\eta(x) \) be defined by formula (13) where \( \gamma_1(x) \) is given by (22).

**THEOREM 3.** Assume (i), (iv), (v) and condition (10). Moreover, suppose that there exists an interval \( J \subseteq I \) such that \( \Gamma_\eta(x) \) tends to zero uniformly on \( J \). Then a regular solution of equation (2) fulfilling the conditions \( \varphi(0) = 0 \) and \( \varphi'(0) = \eta \), depends on an arbitrary function.

**Proof.** On account of our assumptions and by means of Lemma 3 we infer that the assumptions of Theorem 6 from [4] concerning equa-
tion (19) are satisfied. Thus a continuous solution $\psi$ of (19) fulfilling the condition $\psi(0) = \eta$ (if such a solution exists) depends on an arbitrary function. Now, Lemma 4 completes our proof.

Condition

\begin{equation}
K(0, \eta) = \eta
\end{equation}

is necessary for equation (19) to have a continuous solution with $\psi(0) = \eta$.

REMARK 2. The definition of $K(x, z)$ implies that equation (23) has the form

\[ \frac{A}{f(0)} + \frac{B}{f(0)} \eta = \eta. \]

Put

\begin{equation}
K(x) := |K(x, \eta) - \eta| = \begin{cases} \frac{1}{f(x)} g(x, x\eta) - \eta & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases}
\end{equation}

where $\eta$ is a solution of (23) and

\begin{equation}
H_n(x) := \sum_{i=0}^{n-1} \frac{K(f(x))}{f(i+1)} \Gamma_n(x), n = 2, 3, ...
\end{equation}

THEOREM 4. Assume (i), (iv), (v) and condition (10). If, for a fixed $\eta$, there exists a point $x_0 \in \mathbb{N}\setminus\{0\}$ such that both $\Gamma_n(x)$ and $H_n(x)$ tend to zero uniformly on $[f(x_0), x_0]$, then equation (2) has a regular solution $\varphi$ in $I$ fulfilling the conditions $\varphi(0) = 0$ and $\varphi'(0) = \eta$, depending on an arbitrary function.

Proof. The assumptions of our theorem and Lemma 3 imply that the hypotheses of Theorem 7 from [4] concerning equation (19) are satisfied. Consequently, equation (19) has a continuous solution $\psi$ in $I$ fulfilling the condition $\psi(0) = \eta$ and depending on an arbitrary function. Now our assertion results from Lemma 4.

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