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ON CARATHÉODORY TYPE SELECTORS IN A HILBERT SPACE

Abstract. In this paper we consider a set-valued function of two variables, measurable in the first and continuous in the second variable. Using metric projections we construct for this function a family of selectors which are Carathéodory maps. The existence of Carathéodory selectors was studied by Castaing [2], [3], Cellina [4], Fryszkowski [9] and the first author [11].

1. Notation and definitions. Let $(T, \mathscr{T})$ be a measurable space, X a topological space and Y a Hilbert space. By $\mathscr{P}_c(Y)$ we denote the family of all non-empty closed convex subsets of Y. We shall consider $\mathscr{P}_c(Y)$ with the Vietoris topology (see e.g. [12, § 17.1]), and with the *generalized Hausdorff metric*

$$\operatorname{dist}(A, B) = \sup\{d(a, B), d(b, A) : a \in A, b \in B\},$$

where $d(a, B) = \inf\{\|a-b\| : b \in B\}$, $A, B \in \mathscr{P}_c(Y)$ (we admit $\operatorname{dist}(A, B) = \infty$).

Let y_0 be a point of Y and r a positive number. By $B(y_0, r)$ $(\bar{B}(y_0, r))$ we denote the open (closed) ball with centre y_0 and radius r. For a set $A \subset Y$ and $r > 0$, $B(A, r)$ denotes the r-ball about A.

Let $\varphi: T \to \mathscr{P}_c(Y)$ be a multifunction (i.e. set-valued mapping). A function $f: T \to Y$ is a *selector* for φ if $f(t) \in \varphi(t)$ for all $t \in T$. A multifunction φ is *measurable* if

$$\{t \in T : \varphi(t) \cap G \neq \emptyset\} \in \mathscr{T}$$

for each open $G \subset Y$ (such φ is called weakly measurable by Himmelberg [10] and Wagner [15], [16]).

We say $f: T \times X \to Y$ is a *Carathéodory map* if $f(t, \cdot)$ is continuous for each $t \in T$, and $f(\cdot, x)$ is measurable for each $x \in X$.

If A is a non-empty closed convex subset of a Hilbert space Y, then for each $y \in Y$ there is the unique point $h(y, A) \in A$ such that

$$\|y - h(y, A)\| = \inf\{\|y-a\| : a \in A\}$$

(see e.g. [13, Theorem 2.1.2]). The function $h: Y \times \mathscr{P}_c(Y) \to Y$ is called the *metric*

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projection. If $y=0$ then we shall write $h(A)$ instead of $h(0, A)$. By the uniqueness of $h(y, A)$,

$$h(y, A) = h(A-y)+y, \tag{1.1}$$

where $A-y = \{a-y : a \in A\}$.

Let a multifunction $\varphi : T\times X \to \mathscr{P}_c(Y)$ be given. For any function $g : T \to Y$ the mapping $f(t, x) = h(g(t), \varphi(t, x))$ is a selector for φ. The aim of this paper is to formulate conditions under which f is a Carathéodory map. In this way we obtain a family of Carathéodory selectors for φ.

2. Preliminary results. In this section we shall study inverse images of open balls under the metric projection. The following geometric lemma will be useful:

LEMMA 1. *Let Y be a real Hilbert space, $A \in \mathscr{P}_c(Y)$, $y_0 = h(A)$ and $r > 0$. For each $y \in A\setminus B(y_0, r)$,*

$$\|y\|^2 \geqslant \|y_0\|^2 + r^2.$$

Proof. If $0 \in A$ then $y_0 = 0$ and the inequality holds. If $0 \notin A$ then $\langle y_0, y-y_0\rangle \geqslant 0$ for all $y \in A$ ([13, Theorem 2.2.2]). For $y \in A\setminus B(y_0, r)$ we have

$$\|y\|^2 = \|y_0+(y-y_0)\|^2 = \|y_0\|^2 + 2\langle y_0, y-y_0\rangle + \|y-y_0\|^2 \geqslant \|y_0\|^2 + r^2,$$

which completes the proof.

The next lemma will play the key role in the paper.

LEMMA 2. *Let Y be a real Hilbert space, x_0 a point of Y and r a positive number. Then*:

I. $h^{-1}(B(x_0, r)) = \bigcup_q \{A \in \mathscr{P}_c(Y)\colon\ A \subset (Y \setminus \bar{B}(0, q)) \cup B(x_0\, r)$ *and* $A \cap B(0, q) \neq \emptyset\}$,

where the union is taken over all positive q satisfying

$$\|x_0\| - r < q < \|x_0\| + r. \tag{2.1}$$

II. $h^{-1}(B(x_0, r)) = \bigcup_{q,n} \{A \in \mathscr{P}_c(Y) : A \subset (Y\setminus B(0, q)) \cup \bar{B}\left(x_0, r-\frac{1}{n}\right)$ *and* $A \cap B(0, q) \neq \emptyset\}$,

where the union is taken over all positive rationals q and all positive integers n satisfying the following conditions:

$$\|x_0\| - r < q < \|x_0\| + r \text{ and } \frac{1}{n} < r.$$

Proof. We shall prove the equality I. The proof of the second part of the lemma is quite similar, therefore we omit it.

Let $A \in \mathscr{P}_c(Y)$ be such that $y_0 = h(A) \in B(x_0, r)$. We shall show the existence of positive q satisfying (2.1) such that

$$A \subset (Y\setminus \bar{B}(0, q)) \cup B(x_0, r) \text{ and } A \cap B(0, q) \neq \emptyset. \tag{2.2}$$

There is $d > 0$ such that $\bar{B}(y_0, d) \subset B(x_0, r)$, i.e. $\|x_0 - y_0\| + d < r$. It follows from Lemma 1 that

$$\|y\|^2 \geqslant \|y_0\|^2 + d^2 \tag{2.3}$$

for $y \in A \setminus B(y_0, d)$. Let $q > 0$ be such that

$$\|y_0\|^2 < q^2 < \|y_0\|^2 + d^2.$$

It implies $\|x_0\| - r < q < \|x_0\| + r$. Suppose there is $y \in A \cap \bar{B}(0, q)$ such that $y \notin B(y_0, d)$. Because of (2.3), $\|y\| > q$ which is inconsistent with $y \in \bar{B}(0, q)$. Hence, $A \cap \bar{B}(0, q) \subset B(y_0, d)$. Then

$$A = (A \cap (Y \setminus \bar{B}(0, q))) \cup (A \cap \bar{B}(0, q)) \subset (Y \setminus \bar{B}(0, q)) \cup B(y_0, d)$$
$$\subset (Y \setminus \bar{B}(0, q)) \cup B(x_0, r).$$

The intersection of A and $B(0, q)$ is non-empty, because y_0 is a common point of these sets. Thus A satisfies (2.2).

Now assume that $A \in \mathscr{P}_c(Y)$ satisfies (2.2). Since $A \cap B(0, q) \neq \emptyset$, $h(A) \in B(0, q)$. On the other hand, $h(A) \in (Y \setminus \bar{B}(0, q)) \cup B(x_0, r)$. Hence, $h(A) \in B(x_0, r)$, which completes the proof of the first part of the lemma.

3. Continuity of metric projections. In this section we shall study the continuity of the function $h(y, \cdot)$.

THEOREM 1. *Let Y be a real Hilbert space. For each $y \in Y$ the function $h(y, \cdot): \mathscr{P}_c(Y) \to Y$ is continuous in the Vietoris topology and in the generalized Hausdorff metric.*

Proof. Because of (1.1), it suffices to consider the case $y = 0$. The continuity of h in the Vietoris topology is an immediate consequence of Lemma 2.I. Now we show that h is continuous in the generalized Hausdorff metric. Let $A \in \mathscr{P}_c(Y)$ be arbitrary but fixed. Denote $y_0 = h(A)$. We shall prove that for each $r > 0$ there is $s > 0$ such that if $F \in \mathscr{P}_c(Y)$ and $\operatorname{dist}(A, F) < s$, then $h(F) \in B(y_0, r)$. We have to consider two cases:

1°. There is $s > 0$ such that $B(A, s) \subset B(y_0, r)$. If $F \in \mathscr{P}_c(Y)$ and dist $(A, F) < s$, then $F \subset B(A, s)$. Hence, $h(F) \in B(y_0, r)$.

2°. For each $s > 0$, $B(A, s) \setminus B(y_0, r) \neq \emptyset$. Preliminary we shall show the existence of $s > 0$ such that $\|z\| \geq \|y_0\| + s$ for each $z \in B(A, s) \setminus B(y_0, r)$. Fix $0 < s < \frac{r}{2}$ and $z \in B(A, s) \setminus B(y_0, r)$. Let $y \in A$ be such that $\|y - z\| < s$. Since $y \notin B\left(y_0, \frac{r}{2}\right)$, it follows from Lemma 1 that

$$\|y\|^2 \geq \|y_0\|^2 + \frac{r^2}{4}.$$

Thus

$$\|z\| \geq \|y\| - s \geq \sqrt{\|y_0\|^2 + \frac{r^2}{4}} - s.$$

It is not difficult to see that for s satisfying

$$0 < s < \frac{1}{2}\left(\sqrt{\|y_0\|^2 + \frac{r^2}{4}} - \|y_0\|\right)$$

we have

$$\sqrt{\|y_0\|^2 + \frac{r^2}{4}} - s \geq \|y_0\| + s.$$

For such s and $r_1 = \|y_0\| + s$, if $z \in B(A, s) \setminus B(y_0, r)$ then $\|z\| \geqslant r_1$. Let $F \in \mathscr{P}_c(Y)$ be such that $\operatorname{dist}(A, F) < s$. Since $F \subset B(A, s)$, $h(F) \in B(y_0, r) \cup (B(A, s) \setminus B(y_0, r))$. It follows from $A \subset B(F, s)$ that $F \cap B(0, r_1) \neq \emptyset$. Then $\|h(F)\| < r_1$ and, consequently, $h(F) \in B(y_0, r)$. It completes the proof of the continuity of h at A in the generalized Hausdorff metric.

REMARK 1. The Vietoris topology and the topology of the Hausdorff distance coincide on the family of all compact subsets of Y. On $\mathscr{P}_c(Y)$ these two topologies are incomparable. The continuity of metric projections in the Hausdorff metric was studied by several authors (see e.g. Filippov [8, Lemma 5], Daniel [5, Theorem 2.2], Tolstonogov [14, Theorem 1.1]). The corresponding result for the Vietoris topology seems to be new.

4. Measurability of metric projections. Let $(T, \mathscr{T})$ be a measurable space, Y a Hilbert space, $\varphi : T \to \mathscr{P}_c(Y)$ a measurable multifunction, and $g : T \to Y$ a measurable function. In this section we shall prove the measurability of the function $t \to h(g(t), \varphi(t))$, where h is the metric projection.

THEOREM 2. *Let $(T, \mathscr{T})$ be a measurable space, Y a real separable Hilbert space, and $\varphi : T \to \mathscr{P}_c(Y)$ a measurable multifunction. Then for each measurable $g : T \to Y$ the function $t \to h(g(t), \varphi(t))$ is a measurable selector for φ.*

Proof. First we prove that $h(\varphi(\cdot))$ is a measurable function. Let D be a countable dense subset of Y. The family of all balls $B(x, r)$, where $x \in D$ and r is a positive rational, is a countable open base for Y. It suffices to show that inverse images of these balls under h belong to the σ-algebra $\mathscr{T}$. By Lemma 2.II and the measurability of φ, we have

$$\{t \in T : h(\varphi(t)) \in B(x, r)\} = \bigcup_{q,n} \left(\left\{ t \in T : \varphi(t) \subset (Y \setminus B(0, q)) \cup \bar{B}\left(x, r - \frac{1}{n}\right) \right\} \cap \right.$$

$$\left. \cap \{t \in T : \varphi(t) \cap B(0, q) \neq \emptyset\} \right) \in \mathscr{T},$$

where the union is taken over all positive rationals q and all positive integers n satisfying

$$\|x\| - r < q < \|x\| + r \text{ and } \frac{1}{n} < r.$$

Since φ and g are measurable, the multifunction

$$\varphi(t) - g(t) = \{y - g(t) : y \in \varphi(t)\}$$

is also measurable. Because of (1.1),

$$h(g(t), \varphi(t)) = h(\varphi(t) - g(t)) + g(t)$$

and, consequently, the function $t \to h(g(t), \varphi(t))$ is measurable.

REMARK 2. Similar results to Theorem 2 were obtained by Bocşan ([1, Theorem 1]) and Engl and Nashed ([7, Lemma 2.2]) under assumption that the measurable space $(T, \mathscr{T})$ is complete.

5. Carathéodory type selectors. Our main result is an immediate consequence of two previous theorems.

THEOREM 3. *Let* $(T, \mathscr{T})$ *be a measurable space,* X *a topological space,* Y *a real separable Hilbert space, and* $\varphi: T\times X \to \mathscr{P}_c(Y)$ *a multifunction. We assume that for each* $x \in X$, $\varphi(\cdot, x)$ *is measurable and for each* $t \in T$, $\varphi(t, \cdot)$ *is continuous in the Vietoris topology or in the generalized Hausdorff metric. Then for each measurable* $g: T \to Y$ *the function* $f(t, x) = h(g(t), \varphi(t, x))$ *is a Carathéodory selector for* φ.

Proof. It follows from Theorem 2 that $f(\cdot, x)$ is measurable for each $x \in X$. In virtue of Theorem 1, for each $y \in Y$ the function $h(y, \cdot)$ is continuous in the Vietoris topology and in the generalized Hausdorff metric. Thus $f(t, \cdot)$ is continuous as the composition of continuous functions.

REMARKS 3. A multifunction is continuous in the Vietoris topology iff it is lower and upper semi-continuous. For compact-valued multifunctions the continuity in the Vietoris topology is equivalent to the continuity in the Hausdorff distance. These two notions of continuity are incomparable for closed convex-valued multifunctions.

4. Theorem 3 admits the following generalization: Suppose T is endowed with the family of σ-fields $\{\mathscr{T}_x\}_{x\in X}$, for each $x \in X$, $\varphi(\cdot, x)$ is $\mathscr{T}_x$-measurable, and the other assumptions of Theorem 3 are satisfied. If $g: T \to Y$ is measurable with respect to the σ-algebra $\bigcap_{x\in X} \mathscr{T}_x$, then for each $t \in T$ the function $f(t, \cdot)$ is continuous, and for each $x \in X$, $f(\cdot, x)$ is $\mathscr{T}_x$-measurable. The same proof holds. This theorem is of special interest in the case when X is an interval on the real line and $\{\mathscr{T}_x\}_{x\in X}$ is an increasing family of σ-fields. A. Fryszkowski called our attention to the problem of the existence of such "non-anticipative" Carathéodory selectors.

5. We can generalize Theorem 3 in the other way. Suppose for each $t \in T$ the multifunction $\varphi(t, \cdot)$ is defined on a non-empty set $D(t) \subset X$ instead of on the whole space X. In this case $f(t, \cdot)$ is also defined on $D(t)$. We say that such a multifunction φ is measurable in t if for each $x \in X$ and each open $G \subset Y$,

$$\{t\in T : \varphi(t, x) \cap G \neq \emptyset \text{ and } x\in D(t)\} \in \mathscr{T}.$$

In the same way we define the measurability of $f(\cdot, x)$. With this meaning of the measurability, Theorem 3 holds.

6. Under assumptions of Theorem 3 the existence of Carathéodory selectors cannot be deduced from known general results ([2], [3], [9], [11]), because we admit X to be an arbitrary topological space.

7. Ekeland and Valadier [7] used similar methods in the proof of the representation theorem for a multifunction of two variables.

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