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Józef Kalinowski and Jerzy Klamka

POSITIVE DYNAMICAL SYSTEMS WITH DELAYS

Abstract: In the paper using algebraic methods and method of steps new sufficient conditions for positivity of linear continuous-time dynamical systems with variable coefficients and distributed delay and several constant point delays are established. Neutral linear dynamical systems with several constant point delays are also discussed. As a special case linear dynamical systems without delays are also considered. Finally, illustrative example is given.

Key words: linear systems, continuous time systems, positive systems with delays.

1. Introduction

In the paper using algebraic methods and method of steps and results given in monograph [2] and paper [4] sufficient conditions for positivity of linear continuous-time systems with distributed delay and several constant point delays and variable coefficients are formulated and proved. Roughly speaking in the positive dynamical systems trajectory of the system and the initial conditions are non-negative. The results obtained in the present paper generalize to the case of linear dynamical systems with time-variable coefficients and different types of delays, positivity conditions given in the monograph [2] for linear dynamical systems without delays and constant coefficients, and in the paper [4] for linear dynamical systems with variable coefficients and without delays.

Proof of the main result is based on integral formula of the solution and method of steps for the differential state equation. Generally it is shown, that there are important relationships between Metzler matrix and positivity conditions for linear dynamical systems with delays. Moreover, it should be pointed out that the presented results can be extended to cover more general class of dynamical systems with delays e.g., dynamical systems with Lebesque-Stieltjes integral. Finally, simple numerical example, which illustrates theoretical considerations is presented.

2. Preliminaries

Let $\mathbb{R}^{n\times m}$ be the set of $n\times m$ real matrices. The set of $n\times m$ real matrices with non-negative entries will be denoted by $\mathbb{R}_+^{n\times m}$ and $\mathbb{R}_+^n := \mathbb{R}_+^{n\times 1}$.

Consider the continuous-time linear dynamical system with variable coefficients and distributed delay and several constant point delays in the state variables described by the following vector functional differential nonhomogeneous equation

$$(1) \qquad x'(t) = \sum_{k=0}^{M} A_k(t)x(t-h_k) + \int_{-h_M}^{0} A(s)x(t+s)ds + f(t),$$

defined for $t \in [t_0, \infty)$ a.e., where $x(t) \in \mathbb{R}^n$, $A_k(t) \in \mathbb{R}^{n\times n}$, $k = 0, 1, \ldots, M$, the entries $a_{kij} \in L^2_{loc}([t_0, \infty), \mathbb{R})$, $i, j = 1, 2, \ldots, n$ and the function $f \in L^1_{loc}([t_o, \infty), \mathbb{R}^n)$, $0 = h_0 < h_1 < \cdots < h_M$ are constant point delays, $A(s)$ for $s \in [-h_M, 0]$ is $n \times n$ dimensional matrix with continuous entries $a_{ij} \in C([-h_M, 0], \mathbb{R})$ for $i, j = 1, 2, \ldots, n$.

It is well known (see e.g. [1]), that for the dynamical system with delays (1) it is convenient to introduce the following notation: $x_t(s) = x(t+s)$ for $s \in [-h_M, 0]$.

In order to solve vector functional differential equation (1) it is necessary to give initial condition $(x(t_0), x_{t_0}) \in \mathbb{R}^n \times L^2([-h_M, 0], \mathbb{R}^n)$.

From the form of the vector functional differential equation (1) it immediately follows, that for a given initial condition $(x(t_0), x_{t_0})$ and given function $f(t)$ the unique solution $x(t)$ of the equation (1) is absolutely continuous vector function for $t \in [t_0, \infty)$.

In the sequel we shall use the following basic definitions of positive dynamical systems, taken from monograph [2].

Definition 1. ([2]). *The dynamical system (1) is called positive if for all initial conditions $(x(t_0), x_{t_0}) \in \mathbb{R}_+^n \times L^2([-h_M, 0], \mathbb{R}_+^n)$ and arbitrary function $f \in L^1_{loc}([t_0, \infty), \mathbb{R}_+^n)$, we have $x(t) \in \mathbb{R}_+^n$ for $t \geq t_0$.*

Definition 2. ([2]). *A matrix $M \in \mathbb{R}^{n\times n}$ is called the Metzler matrix if its all off-diagonal entries are non-negative.*

It is easy to show (see e.g. monograph [2]) that $\exp(Mt) \in \mathbb{R}_+^{n\times n}$, if and only if M is a Metzler matrix.

In the sequel we shall also consider a special case of the dynamical system (1), i.e., dynamical system without delays described by the following linear vector differential equation with variable coefficients

$$(2) \qquad x'(t) = A_0(t)x(t) + f(t),$$

defined for $t \geq t_0$.

It should be pointed out, that for linear dynamical system (2) with constant coefficients (i.e., $A_0(t) = A_0$) without loss of generality we can take the initial time $t_0 = 0$ (see e.g. [3]).

It is well known that the solution $x(t)$ of vector differential equation (2) with a given initial condition $x(t_0) \in \mathbb{R}^n$ and given function $f \in L^1_{loc}([t_0, \infty), \mathbb{R}^n)$ has the following form (see e.g. [3])

$$x(t) \;=\; F(t, t_0)x(t_0) + \int_{t_0}^{t} F(t, s)f(s)ds \tag{3}$$

for $t \geq t_0$, where $F(t, s)$ is so called state transition matrix [3] given by

$$F(t, t_0) \;=\; I_n + \int_{t_0}^{t} A_0(\tau)d\tau + \int_{t_0}^{t} A_0(\tau) \int_{t_0}^{\tau} A_0(s)dsd\tau + \dots$$

In the special case, when the matrix $A_0(t)$ satisfies the following Lappo-Danilewski additional assumption

$$A_0(t) \cdot \int_{t_0}^{t} A_0(\tau)d\tau = \int_{t_0}^{t} A_0(\tau)d\tau \cdot A_0(t) \quad \text{for} \quad t \in [t_0, \infty)$$

the transition matrix $F(t, t_0)$ is given by

$$F(t, t_0) \;=\; \exp\left(\int_{t_0}^{t} A_0(\tau)d\tau\right).$$

It should be mentioned, that state transition matrix $F(t, \tau)$ is nonsingular for every $t, \tau \in \mathbb{R}$.

Moreover, it should be also pointed out, that for the case when $A_0(t) = A_0$ we have (see [2])

$$F(t, s) \;=\; \exp(A_0(t - s)).$$

Now, we shall recall two lemmas concerning the relationships between Metzler matrix and positive dynamical systems without delays.

Lemma 1. ([4]). *Let us assume that $A_0(t)$ is a Metzler matrix for every $t \geq t_0$. Then the system (2) has the state transition matrix $F(t, t_0) \in \mathbb{R}^{n \times n}_{+}$ for every $t \geq t_0$.*

Lemma 2. ([4]). *The system (2) with $f \in L^1_{loc}([t_0, \infty), \mathbb{R}^n_{+})$ is positive, if $A_0(t)$ is a Metzler matrix for every $t \geq t_0$.*

In the next sections integral formula (3) will be used in the proof of the main results. Let us observe, that for positive linear dynamical systems without delays we have $x(t) \in \mathbb{R}^n_{+}$ for $t \geq t_0$.

3. Main Results

In this section we shall formulate and prove sufficient condition for positivity of the dynamical systems with delays (1).

Theorem 1. *The dynamical system* (1) *is positive if*

1. $A_0(t)$ *is a Metzler matrix for every* $t \geq t_0$,
2. *entries* $a_{kij} \in L^2_{loc}([t_0,\infty),\mathbb{R}_+)$, $k=1,2,\ldots,M$, $i,j=1,2,\ldots,n$,
3. $f \in L^1_{loc}([t_0,\infty),\mathbb{R}^n_+)$,
4. *entries* $a_{ij} \in C([-h_M,0],\mathbb{R}_+)$ *for all* $i,j=1,2,\ldots,n$.

Proof. Proof of the theorem is based on the method of steps. It is well known that the solution $x(t)$ of the functional differential equation (1) for a given initial condition $(x(t_0),x_{t_0}) \in \mathbb{R}^n_+ \times L^1([-h_M,0],\mathbb{R}^n_+)$ has the following form [1], for $t \in (t_0, t_0+h_1]$

$$\begin{aligned} x(t) &= F(t,t_0)\,x(t_0) \\ &+ \int_{t_0}^{t} F(t,s)\left[\sum_{k=0}^{M} A_k(s)x_{t_0}(s-h_k) + \int_{-h_M}^{0} A(\tau)x(\tau+s)d\tau + f(s)\right] ds. \end{aligned}$$

Let the symbol $W^{1,2}([t_0,t_0+h_1],\mathbb{R}^n_+)$ denotes the Sobolev space of nonnegative vector functions defined in the interval $[t_0,t_0+h_1]$, which have square integrable first order derivative [3]. Since $A_0(t)$ is a Metzler matrix for $t \in [t_0,t_0+h_1]$ and

$$\sum_{k=0}^{M} A_k(s)x_0(s-h_k) + \int_{-h_M}^{0} A(\tau)x(\tau+s)d\tau + f(s) \in W^{1,2}([t_0,t_0+h_1],\mathbb{R}^n_+)$$

then $x(t) \in \mathbb{R}^n_+$ for $t \in [t_0,t_0+h_1]$. Hence $x(t_0+h_1) \in \mathbb{R}^n_+$.

Similarly, using the method of steps, it can be shown that for any $k = 2,3,\ldots,M$, we have $x(t_0+h_k) \in \mathbb{R}^n_+$. It directly follows from the equalities:

a) for $t \in (t_0+h_1, t_0+h_2]$

$$\begin{aligned} x(t) &= F(t,t_0+h_1)\,x(t_0+h_1) \\ &+ \int_{t_0+h_1}^{t} F(t,s)\left[\sum_{k=0}^{M} A_k(s)x_{t_0+h_1}(s-h_k)\right. \\ &\qquad\qquad \left. + \int_{-h_M}^{0} A(\tau)x(\tau+s)d\tau + f(s)\right] ds, \end{aligned}$$

b) for $t \in (t_0 + h_k, t_0 + h_{k+1}]$,

$$\begin{aligned} x(t) &= F(t, t_0 + h_k)\, x(t_0 + h_k) \\ &\quad + \int_{t_0+h_k}^{t} F(t,s) \left[\sum_{k=0}^{M} A_k(s) x_{t_0+h_k}(s - h_k) \right. \\ &\quad \left. + \int_{-h_M}^{0} A(\tau) x(\tau + s) d\tau + f(s) \right] ds, \end{aligned}$$

c) for $t \in (t_0 + h_M, \infty)$,

$$\begin{aligned} x(t) &= F(t, t_0 + h_M)\, x(t_0 + h_M) \\ &\quad + \int_{t_0+h_M}^{t} F(t,s) \left[\sum_{k=0}^{M} A_k(s) x_{t_0+h_M}(s - h_k) \right. \\ &\quad \left. + \int_{-h_M}^{0} A(\tau) x(\tau + s) d\tau + f(s) \right] ds. \end{aligned}$$

Therefore, $x(t) \in \mathbb{R}^n_+$ for $t \geq t_0$ and Theorem 1 is proved. ■

Now, let us discuss special case of dynamical system (1) i.e., linear dynamical system without distributed delay but with only several point delays described by the following linear vector differential equation

$$x'(t) = \sum_{k=0}^{M} A_k(t) x(t - h_k) + f(t). \tag{4}$$

Corollary 1. *The dynamical system* (4) *is positive if*

1. $A_0(t)$ is a Metzler matrix for every $t \geq t_0$,

2. $a_{kij} \in L^2_{loc}([t_0, \infty), \mathbb{R}_+)$, $k = 1, 2, \ldots, M$, $i, j = 1, 2, \ldots, n$,

3. $f \in L^1_{loc}([t_0, \infty), \mathbb{R}^n_+)$.

From Theorem 1 it directly follows sufficient condition for positivity of the linear dynamical system with variable coefficients and without delays.

Corollary 2. ([3]). *The dynamical system without delays* (2) *is positive if*

1. $A_0(t)$ is a Metzler matrix for every $t \geq t_0$,

2. $f \in L^1_{loc}([t_0, \infty), \mathbb{R}^n_+)$.

The next corollary proved in the monograph [2] gives necessary and sufficient condition for positivity of the linear dynamical systems without delays and with constant coefficients.

Corollary 3. ([2]). *The dynamical system without delays* (2) *and with constant coefficients (i.e., $A_0(t) = A_0$) is positive if and only if*

1. *A_0 is a Metzler matrix,*
2. *$f \in L^1_{loc}([0,\infty), \mathbb{R}^n_+)$.*

Finally, let us consider linear vector neutral differential equation with variable coefficients and several constant point delays of the following form

$$x'(t) = \sum_{k=0}^{M} A_k(t)x(t-h_k) + \sum_{k=1}^{M} C_k(t)x'(t-h_k) + f(t), \tag{5}$$

where $C_k(t)$, $k = 1,2,\ldots,M$ are $n \times n$ dimensional matrices with entries $c_{kij} \in C^1([-h_M, 0], \mathbb{R})$. For dynamical system (5) the initial condition $x_{t_0} \in C^1([-h_M, 0], \mathbb{R}^n_+)$.

Since $x(t)$, $t > 0$, depends on the past values of the derivatives $x'(t-h_k)$, then simple observation shows, that in this case even nonnegativity of all matrices $C_k(t)$, $k = 1,2,\ldots,M$ does not ensure positivity of the solution $x(t)$ for $t > 0$.

The next simple numerical example illustrates the essential role of the assumption 2 in Theorem 1.

Example 1. For $n = 2$, $M = 1$, $t_0 = 0$ and $h_1 = 1$ consider the homogeneous linear delay system (1) with Metzler matrices A_0 and A_1 of the following form

$$x'(t) = \begin{bmatrix} -1 & 0 \\ a & -1 \end{bmatrix} x(t) + \begin{bmatrix} -5 & 0 \\ b & c \end{bmatrix} x(t-1) \quad \text{for} \quad t \geq 0,$$

where $a > 0$, $b > 0$ and $c \in \mathbb{R}$ are constants.

The initial function $x_0(t)$ and initial vector x_0 are given by

$$x_0(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for } t \in [-1, 0]; \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let us observe, that variable $x_1(t) = e^{-t} - 5$ for $t \geq 0$ and $x_1(1) < 0$. Then, dynamical system (1) is not positive. Hence, the assumption, that A_1 is a Metzler matrix does not ensure positivity of the system (1). However, it should be pointed out, that for positivity of the dynamical system (1) it is necessary to assume $A_1 \in \mathbb{R}^{2\times 2}_+$.

4. Conclusions

In the paper using algebraic methods and method of steps sufficient conditions for positivity of linear continuous-time systems with distributed delay and several constant point delays and variable coefficients have been formulated and proved. The results obtained in the present paper generalize to the case of dynamical systems with time-variable coefficients and different types of delays, positivity conditions given in the monograph [2] and in the paper [4]. Monograph [2] contains positivity results for linear dynamical systems without delays and constant coefficients and paper [4] discusses positivity property for linear dynamical system with variable coefficients and without delays.

Finally, it should be pointed out that using quite similar algebraic methods based on Metzler matrix and generalized permutation matrix it is possible to formulate and to prove controllability and reachability results (see e.g., [3] for more details) for linear positive dynamical systems with several point delays in control or in the state variables.

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Józef Kalinowski
Department of Mathematics, Silesian University
Bankowa 14, 40–007 Katowice, Poland
e-mail: kalinows@ux2.math.us.edu.pl

Jerzy Klamka
Department of Automation, Silesian Technical University
Akademicka 16, 44-100 Gliwice, Poland
e-mail: jklamka@ia.polsl.gliwice.pl

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