Title: Rigid graphs of maps

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RIGID GRAPHS OF MAPS

Abstract. In this note we construct maps between metric separable connected spaces $X$ and $Y$ such that the graphs are connected, dense and rigid subspaces of the Cartesian product $X \times Y$. From this result it follows that there is no maximal topology among metric separable connected topologies on a given set $X$.

In this note we shall construct maps between metric separable connected spaces $X$ and $Y$ such that the graphs are connected, dense and rigid subspaces of the Cartesian product $X \times Y$. The first construction of a map $f: \mathbb{R} \to \mathbb{R}$ with the connected and dense graph in the plane and satisfying the Cauchy equation $f(x) + f(y) = f(x+y)$ was given by F.B. Jones [3] in 1942. More general construction one can find in [4].

In order to obtain the existence of rigid graphs of maps, we shall utilize, in the proof, an idea of W. Sierpiński from [5]. A similar method is also used in de Groot's paper [2].

Spaces considered here are assumed to be separable and metric, i.e. we assume that they are subspaces of the Hilbert's cube $I^\omega$.

A continuous map $f: X \to Y$, $X, Y \subset I^\omega$, is called a continuous displacement [2], iff there exists a subset $V \subset X$ such that

$$|f(V)| = 2^\omega$$

and $V \cap f(V) = \emptyset$.

Let us notice that each homeomorphism $f: X \to X$ different from the identity map, and where $X$ is a connected subspace of $I^\omega$, ia a continuous displacement. Indeed, since $f \neq \text{id}_X$, there exists a point $x \in X$ such that $f(x) \neq x$. Choose disjoint open sets $V, W \subset X$ such that $x \in V$ and $f(x) \in f(V) \subset W$. Since $X$ is a connected metric space hence $|V| = 2^\omega$. Thus, $|f(V)| = 2^\omega$ and $V \cap f(V) = \emptyset$.

For more exhaustive information on continuous displacements, the reader can refer to de Groot's paper [2].

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A space $X$ is said to be *rigid* if it admits between itself no homeomorphism different from the identity map. An abundant information on rigid spaces can be found in Charatonik's paper [1].

For each map $f : X \to Y$, let $G(f)$ denotes the graph of the map $f$: $$G(f) = \{(x, y) \in X \times Y : y = f(x)\}.$$ Let $\pi : X \times Y \to X$ means the projection and let the symbols $\text{Int}$, $\text{Bd}$ mean respectively interior and boundary operations.

Let us start from a

**Lemma.** If $f : X \to Y$ is a map between connected metric separable spaces such that for each non-empty open set $G \subset X \times Y$ with non-empty boundary $$G(f) \cap \text{Bd}_{X \times Y} G \neq \emptyset,$$

then the graph is connected and dense in $X \times Y$.

**Proof.** It is obvious that the graph must be dense in $X \times Y$, because the sets of the form $U \times V$, $U$ open in $X$ and $V$ open in $Y$, create a base for the topology of the space $X \times Y$.

In order to see that the graph must be connected we shall utilize two results from [4]. It was proved in ([4, Lemma 1]) that if $X$ and $Y$ are connected spaces and $G$ is a non-empty subset of $X \times Y$ then one of the following conditions is satisfied:

(a) $\text{Int}_X \pi (\text{Bd}_{X \times Y} G) \neq \emptyset$,
(b) there exists an $x \in X$ such that $\pi^{-1}(x) \subset \text{Bd}_{X \times Y} G$,
(c) $G$ is dense in $X \times Y$.

Secondly ([4, Lemma 2]), if $D$ is a dense subset of a connected space $Z$ such that for each non-empty open set $G \subset Z$ with $Z \neq G$,

$$D \cap \text{Bd}_Z G \neq \emptyset$$

then $D$ is a connected set.

Put $D = G(f)$ and $Z = X \times Y$. Let us verify that the condition $D \cap \text{Bd}_Z G \neq \emptyset$ is satisfied for each non-empty open set $G \subset Z$ for that $D \subset G$.

1. If $\text{Int}_X \pi (\text{Bd}_Z G) \neq \emptyset$ then according to the assumption $D \cap \text{Bd}_Z G \neq \emptyset$.
2. If there exists an $x \in X$ such that $\pi^{-1}(x) \subset \text{Bd}_Z G$ then it is clear that $D \cap \text{Bd}_Z G \neq \emptyset$.
3. If $G \subset D$ is dense in $Z$ then $$D \cap \text{Bd}_Z G = D \cap (Z \setminus G) = D \setminus G \neq \emptyset.$$

Thus, the lemma is proved.

**Theorem.** Let $X$ and $Y$ be metric separable and connected spaces. Then there exists a family $\mathcal{C} \subset \text{Map}(X, Y)$, $|\mathcal{C}| = 2^c$, $c = 2^c$, such that:

1. Each graph $G(f), f \in \mathcal{C}$, is a connected, dense and rigid subspace of the product $X \times Y$,
2. No two distinct graphs $G(f)$ and $G(g)$, $f, g \in \mathcal{C}$, are homeomorphic.
Proof. Assume that the product $X \times Y$ is a subspace of the Hilbert cube $I^\omega$, $X \times Y \subset I^\omega$. Consider the family

$$\{(f_\alpha : S_\alpha \to I^\omega) : \alpha < 2^\omega\}$$

of all the continuous displacement $f_\alpha : S_\alpha \to I^\omega$, where $S_\alpha$ is a $\emptyset_\beta$ subset of $I^\omega$, such that

$$\pi[S_\alpha \cap (X \times Y)] = 2^\omega,$$

where $\pi : X \times Y \to X$ is the projection. Let us well order the set $X$;

$$X = \{x_\alpha : \alpha < 2^\omega\}$$

and let us put, for each $\alpha < 2^\omega$, $Q_\alpha = \{x_\alpha\} \times Y$. Let $\{P_\alpha : \alpha < 2^\omega\}$ be a well-ordering of the family

$$\{\text{Bd}_{X \times Y} G : G \text{ is open in } X \times Y \text{ and } \text{Int}_X \pi(\text{Bd}_{X \times Y} G) \neq \emptyset\}.$$

We shall define by induction sets

$$A_\alpha = \{p_\alpha, q_\alpha, r_\alpha, s_\alpha, t_\alpha\} \subset X \times Y, \alpha < 2^\omega,$$

satisfying the following conditions:

1. $p_\alpha \in P_\alpha, q_\alpha \in Q_\alpha, r_\alpha, s_\alpha \in S_\alpha \cap (X \times Y), s_\alpha \neq t_\alpha$ and $\pi(s_\alpha) = \pi(t_\alpha)$,
2. if $x, y \in \bigcup \{A_\alpha \setminus \{x_\alpha\} : \alpha < 2^\omega\}$ and $x \neq y$ then $\pi(x) \neq \pi(y)$,
3. for each $\alpha < 2^\omega$, $f_\beta(r_\beta) \notin \bigcup \{A_\beta : \beta < 2^\omega\}$.

Suppose that the sets $A_\beta$ have been chosen for each $\beta < \alpha$. Put

$$Z_\alpha = \bigcup \{A_\beta : \beta < \alpha\}.$$

We have $|Z_\alpha| < 2^\omega$.

(a) Let us choose a $p_\alpha \in P_\alpha$ such that

$$p_\alpha \in P_\alpha \setminus \{f_\beta(r_\beta) : \beta < \alpha\} \text{ and } \pi(p_\alpha) \notin \pi(Z_\alpha).$$

(b) Choose a $q_\alpha \in Q_\alpha$ such that

$$q_\alpha = q_0 \text{ whenever } Q_\alpha \cap (Z_\alpha \cup \{p_\alpha\}) \neq \emptyset$$

or

$$q_\alpha \in Q_\alpha \setminus \{f_\beta(r_\beta) : \beta < \alpha\} \text{ whenever } Q_\alpha \cap (Z_\alpha \cup \{p_\alpha\}) = \emptyset.$$

(c) Let $V_\alpha \subset S_\alpha$ be a set such that

$$|f_\alpha(V_\alpha)| = 2^\omega \text{ and } V_\alpha \cap f_\alpha(V_\alpha) = \emptyset.$$

Choose points $r_\alpha, s_\alpha \in S_\alpha \cap (X \times Y)$ such that

$$r_\alpha, s_\alpha \in f_\alpha^{-1}[f_\alpha(V_\alpha) \setminus (Z_\alpha \cup \{p_\alpha, q_\alpha\})] \setminus \{f_\beta(r_\beta) : \beta < \alpha\},$$

$$\pi(r_\alpha) \neq \pi(s_\alpha) \text{ and } \pi(r_\alpha), \pi(s_\alpha) \notin \pi(Z_\alpha \cup \{p_\alpha, q_\alpha\}).$$

(d) Finally, choose $t_\alpha \in X \times Y$ such that

$$t_\alpha \in \pi(s_\alpha) \times Y \setminus \{f_\beta(r_\beta) : \beta \leq \alpha\}.$$

One can verify that the conditions (a)–(d) imply the conditions (1)–(3).

Let us put $S = \{s_\alpha : \alpha < 2^\omega\}$. The set $S$ can be represented as the union

$$S = \bigcup \{B_\gamma : \gamma < 2^\omega\}, \epsilon = 2^\omega,$$

such that

$$\gamma \neq \gamma' \text{ implies } B_\gamma \neq B_{\gamma'}.$$
Define for each \( \gamma < 2^c \) the set
\[
K_\gamma = \bigcup \{ \{ p_\alpha, q_\alpha, r_\alpha, d_\alpha^\gamma \} : \alpha < 2^\omega \},
\]
where
\[
d_\alpha^\gamma = \begin{cases} s_\alpha, & \text{if } s_\alpha \in B_\gamma, \\ t_\alpha, & \text{if } s_\alpha \notin B_\gamma. \end{cases}
\]
Let \( g_\gamma : X \to Y \) be such that \( G(g) = K_\gamma \).
Since each set \( K \) contains the set \( \bigcup \{ p_\alpha, q_\alpha : \alpha < 2^\omega \} \) hence according to Lemma each of the sets, \( K_\gamma < 2^c \), is dense and connected in the product \( X \times Y \).

Now, suppose that there exists a continuous displacement \( f : K_\gamma \to K_\gamma ', \gamma, \gamma' < 2^c \).
Since \( K_\gamma \subset X \times Y \subset I^\omega \), we can consider the map \( f \) as a continuous displacement \( f : K_\gamma \to I^\omega \).

By Lavrientieff’s Theorem there exists a continuous extension of \( f, f^* : K_\gamma^* \to I^\omega \), where \( K_\gamma^* \supseteq K_\gamma \) is a \( G_\delta \) subspace of \( I^\omega \). According to the construction there exists an \( \alpha < 2^\omega \) such that
\[
f^* = f_\alpha \text{ and } S_\alpha = K_\gamma^*.
\]
Consider the point \( r_\alpha \in S_\alpha \). By the construction we get
\[
r_\alpha \in S_\alpha \cap K_\gamma \text{ and } f_\alpha (r_\alpha) \notin K_\gamma', \text{ for each } \gamma' < 2^c.
\]
Hence
\[
f(r_\alpha) = f^* (r_\alpha) = f_\alpha (r_\alpha) \notin K_\gamma',
\]
that contradicts with \( f(r_\alpha) \in K_\gamma' \).

**COROLLARY.** There exist \( 2^c \) non-homeomorphic, connected rigid subspaces of the Hilbert cube \( I^\omega \).

If we put in Theorem \( X = Y = R \) then we get

**COROLLARY.** On the set of reals, there exist \( 2^c \) non-homeomorphic metric connected separable and rigid topologies which are finer than the natural topology of the space \( R \) of reals.

**COROLLARY.** There is no maximal topology among metric separable connected topologies on the set \( X \).

**Proof.** Suppose that \( X \) is a maximal connected metric separable space. Let \( f : X \to Y \) be a map such that the graph \( G(f) \subset X \times Y \) is a rigid connected and dense subspace of the product \( X \times Y \). The projection \( \pi : G(f) \onto X \) induces a topology on the set \( X \) which is finer than the previous topology.

**REFERENCES**