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# A NOTE ON REMAINDERS OF COMPACT EXTENSIONS

**Abstract.** The paper contains a construction of a Tychonoff space $X$ such that for every compact extension $bX$ the subset $bX-X$ contains a non-empty $\mathcal{G}_\delta$-set $G$ such that $\operatorname{Int} G=\emptyset$.

In 1960 Fine and Gillman [4] proved that if $X$ is a locally compact real compact space, then the remainder of the Čech-Stone compactification of $X$ (abbreviated $\beta X-X$) has the following property: every non-empty $\mathcal{G}_\delta$-set contains a non--empty open set. Hausdorff spaces satisfying this property are called *$P'$-spaces,* whereas *$P$-spaces* are the spaces in which all $\mathcal{G}_\delta$-sets are open; see e.g. Gillman and Jerison [5], Plank [6] or Veksler [7]. Although every compact (Hausdorff) $P$-space is finite, there exist non-trivial compact $P'$-spaces. As an example of a non-trivial compact $P'$-space one can state $\beta N-N$, the remainder of the Čech-Stone compactification of the integers.

Recently Aniskovič [1] has shown that the result of Fine and Gillman can be improved by replacing the Čech-Stone compactification by a wide class of compactifications. He has also pointed out that by an additional set-theoretical assumption one can construct a Tychonoff space no compactification of which has the remainder being a $P'$-space. The aim of this note is to construct such spaces without any additional set-theoretical assumptions.

LEMMA 1. *Every countable $P'$-space is discrete.*

Proof. Indeed, in countable Hausdorff spaces every point is a $\mathcal{G}_\delta$-set.

LEMMA 2. *Every uncountable $P'$-space is non-separable.*

Proof. Let $D$ be a countable subset of an uncountable $P'$-space $X$. Choose a point $x$ of $X-D$. There exists a $\mathcal{G}_\delta$-set $G$ such that $x\in G\subset X-D$. Since $\operatorname{Int}G\neq\emptyset$, $D$ is not dense in $X$.

A topological space $E$ is *extremally disconnected* (abbreviated e.d.) if the closure of every open subset of $E$ is open. Clearly, dense subspaces of e.d. spaces are e.d. For every topological space $X$ there exists so called *absolute* (or *Gleason space*) of $X$, that is an e.d. space $G(X)$ which can be mapped onto $X$ by a continuous

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irreducible perfect mapping; see e.g. Comfort and Negrepontis [2] for the compact case.

THEOREM 1. *There exists an e.d. locally compact space $X$ such that for every compactification $bX$ of the space $X$, the remainder $bX-X$ is either finite or is not a $P'$-space.*

Proof. Let $E$ be an e.d. compact space without isolated points (e.g. the absolute of the Cantor set). Choose a countable discrete subset $N$ of $E$. Clearly, $\mathrm{cl}N$ is a nowhere dense subset of $E$. Thus, $E$ is a compactification of the e.d. locally compact space $X=E-\mathrm{cl}N$. By a theorem of Taĭmanov (see e.g. Engelking [3, p. 182]), every continuous mapping of $X$ into a compact space has a continuous extension over $E$. Thus, $E$ is equivalent to $\beta X$. Now, let $bX$ be an arbitrary compactification of $X$. Then, there exists a continuous mapping $f$ from $E$ onto $bX$ such that $f(\mathrm{cl}N)=$ $=bX-X$. Hence $bX-X$ is a separable compact space. Assume that it is a $P'$-space. Then, by Lemma 2 and Lemma 1, it must be finite.

In particular, Theorem 1 says that in the Theorem of Fine and Gillman mentioned above the assumption of realcompactness cannot be removed.

A Tychonoff space is called *nowhere locally compact* whenever every compact subset of this space is nowhere dense.

THEOREM 2. *There exists an e.d. nowhere locally compact space no compactification of which has the remainder being a $P'$-space.*

Proof. Let $E$ be the absolute of the Cantor set and let $D$ be a countable dense subset of $E$. We set $X=E-D$. By the Taĭmanov's Theorem (see the proof of Theorem 1), $E$ is a compactification of $X$ equivalent to $\beta X$. Since $D$ is countable, the remainder of any compactification of $X$ is countable. Furthermore, $X$ is nowhere locally compact because $D$ is dense in $E$. Suppose $bX$ is a compactification of $X$ with $bX-X$ being a $P'$-space. By Lemma 1, $bX-X$ is discrete. Thus, $X$ is locally compact in some points; we get a contradiction.

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