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To the memory of Professor Andrzej Lasota

Zygfryd Kominek

REMARKS ON THE STABILITY OF SOME QUADRATIC FUNCTIONAL EQUATIONS

Abstract. Stability problems concerning the functional equations of the form

$$f(2x+y) = 4f(x) + f(y) + f(x+y) - f(x-y),$$

and

$$f(2x+y) + f(2x-y) = 8f(x) + 2f(y)$$

are investigated. We prove that if the norm of the difference between the LHS and the RHS of one of equations (1) or (2), calculated for a function g is say, dominated by a function φ in two variables having some standard properties then there exists a unique solution f of this equation and the norm of the difference between g and f is controlled by a function depending on φ.

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1. INTRODUCTION

In paper [8], C. Park and J. Su An considered the following functional equations

$$f(2x+y) = 4f(x) + f(y) + f(x+y) - f(x-y) \tag{1}$$

and

$$f(2x+y) + f(2x-y) = 8f(x) + 2f(y) \tag{2}$$

in the class of functions transforming a real linear space X into another real linear space Y. They have proved that any of equations (1) and (2) and the quadratic functional equation

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \qquad x, y \in X \tag{3}$$

are equivalent. Stability of equation (3) was widely considered (cf., e.g., [2, 3, 5, 6]). In the case of Y a Banach space, the authors of [8] also considered the problem of

Hyers-Ulam-Rassias stability (see [4]) of equations (1) and (2). In particular, they have proved that if $\alpha \in (0,2)$ is a constant and

$$\|g(2x+y) - 4g(x) - g(y) - g(x+y) + g(x-y)\| \leq \theta(\|x\|^\alpha + \|y\|^\alpha), \tag{4}$$

for all $x, y \in X$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|g(x) - Q(x)\| \leq \frac{\theta}{|2^\alpha - 4|}\|x\|^\alpha, \quad x \in X.$$

In this note we will show that equations (1), (2) and (3) are equivalent in a more general case. We will also show that in the stability results one may replace the right-hand-side of (4) by a function φ in two variables having some natural properties (cf. [1,6]). In particular, we cover the case of inequality (4) with $\alpha \neq 2$. However, we obtain somewhat larger estimation constant.

2. EQUIVALENCE OF EQUATIONS (1), (2) AND (3)

Theorem 1. *Let X and Y be commutative groups, the latter without elements of order two. Then in the class of functions transforming X into Y, equations* (1), (2) *and* (3) *are equivalent.*

Proof. Assume that $f : X \to Y$ is a solution of equation (1). Putting $x = y = 0$ in (1) we obtain $4f(0) = 0$, whence $f(0) = 0$. Setting $y = 0$ in (1), we get $f(2x) = 4f(x)$ and for $x = 0$ it follows from (1) that $f(y) = f(-y)$. For arbitrary $x, y \in X$, there is

$$\begin{aligned}4f(x+y) + 4f(x-y) &= f(2x+2y) + f(2x-2y) = \\ &= 4f(x) + f(2y) + f(x+2y) - f(x-2y) + \\ &\quad + 4f(x) + f(2y) + f(x-2y) - f(x+2y) = \\ &= 8f(x) + 8f(y),\end{aligned}$$

which means that f satisfies equation (3).
Assume that Q is a solution of equation (3). Then Q is even and

$$\begin{aligned}Q(2x+y) + Q(-y) &= Q(x + (x+y)) + Q(x - (x+y)) = \\ &= 2Q(x) + 2Q(x+y) = \\ &= 2Q(x) + Q(x+y) + [2Q(x) + 2Q(y) - Q(x-y)] = \\ &= 4Q(x) + Q(x+y) + 2Q(y) - Q(x-y).\end{aligned}$$

Therefore, Q fulfils equation (1).

Assume that f satisfies (2). Setting $x = y = 0$, we get $8f(0) = 0$ and hence $f(0) = 0$. If $y = 0$, then $f(2x) = 4f(x)$ and if $x = 0$, then $f(-y) = f(y)$. Therefore,

$$\begin{aligned}4f(x+y) + 4f(x-y) &= f(2x+2y) + f(2x-2y) = 8f(x) + 2f(2y) = \\ &= 8f(x) + 8f(y),\end{aligned}$$

which means that f satisfies (3).

Now assume that Q satisfies (3). Then $Q(0) = 0$, $Q(y) = Q(-y)$ and $Q(2x) = 4Q(x)$. Thus

$$Q(2x + y) + Q(2x - y) = 2Q(2x) + 2Q(y) = 8Q(x) + 2Q(y),$$

which ends the proof. □

Remark 1. *The assumption that Y has no elements of order two is essential. To see this, consider the group $\{0, 1, 2, 3, 4, 5, 6, 7\}$ with the usual addition mod|8. Then $f \equiv 6$ is a solution of (1) but it does not satisfy equation (3). Moreover, $f \equiv 1$ is a solution of (2) but it satisfies neither equation (1) nor (3).*

3. GENERAL LEMMA ON STABILITY

In the rest of the paper we assume that:

— X is a commutative group,
— Y is a real Banach space.

Moreover, we use the convention:

— $X^\star = X \setminus \{0\}$;
— If not stated otherwise, any formula containing variables x and/or y is valid for all $x, y \in X^\star$.

We start with some lemmas.

Lemma 1. *Let $g : X \to Y$ be a function satisfying the inequality*

$$\|\sum_{i=1}^{r} \alpha_i g(\gamma_i x + \delta_i y)\| \le \varphi(x, y), \tag{5}$$

where we are given: a positive integer r, real constants α_i, integer constants $\gamma_i, \delta_i, i \in \{1, \dots, r\}$ such that $\gamma_i \delta_i \ne 0$ for some $i \in \{1, \dots, r\}$ and $\delta_i \gamma_j \ne \delta_j \gamma_i$ for every $j \ne i, j \in \{1, \dots, r\}$, real constants $\lambda_n, n \in \mathbb{N}, \lambda_0 = 1$, integer constants $\beta_n, n \in \mathbb{N}, \beta_0 = 1$, while $\varphi : X^\star \times X^\star \to [0, \infty)$ is a function satisfying the conditions

$$\begin{cases} \lim_{n\to\infty} \lambda_n \varphi(\beta_n x, \beta_n y) = 0; \\ \sum_{n=0}^{\infty} \lambda_n \varphi(\beta_n x, \beta_n x) < \infty. \end{cases} \tag{6}$$

If there exists a constant $K > 0$ such that

$$\|\lambda_{n+1} g(\beta_{n+1} x) - \lambda_n g(\beta_n x)\| \le K \lambda_n \varphi(\beta_n x, \beta_n x), n \in \mathbb{N} \cup \{0\}, \tag{7}$$

then for every $x \in X$ the sequence $(\lambda_n g(\beta_n x))_{n\in\mathbb{N}}$ converges to a function $f : X \to Y$ fulfilling the equation

$$\sum_{i=1}^{r} \alpha_i f(\gamma_i x + \delta_i y) = 0 \tag{8}$$

and the estimate

$$\|g(x) - f(x)\| \le K \sum_{n=0}^{\infty} \lambda_n \varphi(\beta_n x, \beta_n x). \tag{9}$$

Proof. It follows from (7) that for given positive integers n, k there is

$$\begin{aligned}\|\lambda_{n+k} g(\beta_{n+k} x) - \lambda_n g(\beta_n x)\| &\le \sum_{j=0}^{k-1} \|\lambda_{n+j+1} g(\beta_{n+j+1} x) - \lambda_{n+j} g(\beta_{n+j} x)\| \le \\ &\le K \sum_{j=n}^{n+k-1} \lambda_j \varphi(\beta_j x, \beta_j x).\end{aligned}$$

Therefore $(\lambda_n g(\beta_n x))_{n\in\mathbb{N}}$ is a Cauchy sequence, whence it is convergent. Define a function $f^\star : X^\star \to Y$ by the equality

$$f^\star(x) = \lim_{n\to\infty} \lambda_n g(\beta_n x).$$

In (5) let us put $\beta_n x$ instead of x, $\beta_n y$ instead of y and multiply both sides of (5) by λ_n. On account of (6), taking the limit as n tends to infinity, we obtain

$$\sum_{i=1}^{r} \alpha_i f(\gamma_i x + \delta_i y) = 0,$$

where

$$f(x) = \begin{cases} f^\star(x), & x \in X^\star, \\ \lim_{n\to\infty} \lambda_n g(0), & x = 0. \end{cases}$$

Moreover,

$$\begin{aligned}\|g(x) - \lambda_n g(\beta_n x)\| &\le \sum_{k=0}^{n-1} \|\lambda_{k+1} g(\beta_{k+1} x) - \lambda_k g(\beta_k x)\| \le \\ &\le K \sum_{k=0}^{\infty} \lambda_k \varphi(\beta_k x, \beta_k x).\end{aligned}$$

As n tends to infinity, we get

$$\|g(x) - f(x)\| \le K \sum_{n=0}^{\infty} \lambda_n \varphi(\beta_n x, \beta_n x).$$

□

4. LEMMAS ON EQUATIONS (1) AND (2)

Lemma 2. *If $f : X \to Y$ satisfies* (1) *for all $x, y \in X^\star$, then it satisfies* (1) *for all $x, y \in X$.*

Proof. Setting, successively, $y = x, y = 2x, y = -x, x \neq 0$ in (1), we get

$$f(3x) = 5f(x) + f(2x) - f(0), \tag{10}$$
$$f(4x) = 4f(x) + f(2x) + f(3x) - f(-x), \tag{11}$$
$$f(2x) = 3f(x) + f(-x) + f(0). \tag{12}$$

Adding (10) and (11), we obtain

$$f(4x) = 9f(x) + 2f(2x) - f(-x) - f(0),$$

and, thanks to (12),

$$3f(2x) + f(-2x) + f(0) = 9f(x) + 2f(2x) - f(-x) - f(0).$$

Applying (12) once more, we observe that

$$3f(x) + f(-x) + f(0) + 3f(-x) + f(x) + f(0) + f(0) = 9f(x) - f(-x) - f(0),$$

which implies that

$$4f(0) = 5[f(x) - f(-x)].$$

Consequently, $f(0) = 0, f(-x) = f(x)$ and $f(2x) = 2f(x)$. Now it is easy to verify that (1) is fulfilled for all $x, y \in X$. □

Lemma 3. *If $f : X \to Y$ satisfies* (2) *for all $x, y \in X^\star$, then it satisfies* (2) *for all $x, y \in X$.*

Proof. Setting $y = x$, $x \neq 0$ in (2), we get

$$f(3x) = 9f(x), \tag{13}$$

and the substitution $y = -x, x \neq 0$ in (2) yields $f(3x) = 7f(x) + 2f(-x)$, whence

$$f(x) = f(-x).$$

If we put, successively, $y = 2x, y = 4x, x \neq 0$ in (2), then

$$f(4x) + f(0) = 8f(x) + 2f(2x), \quad f(6x) + f(2x) = 8f(x) + 2f(4x). \tag{14}$$

On account of (14) and (13) we obtain

$$9f(2x) + f(2x) = 8f(x) + 16f(x) + 4f(2x) - 2f(0)$$

whence

$$f(2x) = 4f(x) - \frac{1}{3}f(0). \tag{15}$$

According to (15) and (13),

$$f(6x) = 4f(3x) - \frac{1}{3}f(0) = 36f(x) - \frac{1}{3}f(0).$$

On the other hand, by virtue of (14) and (15), we get

$$36f(x) - \frac{1}{3}f(0) + f(2x) = 8f(x) + 2[8f(x) + 2f(2x) - f(0)]$$

whence

$$f(0) = 0 \qquad \text{and} \qquad f(2x) = 4f(x),$$

by (15). Using these equalities together with $f(x) = f(-x)$, one can easily verify that (2) is fulfilled for all $x, y \in X$. $\square$

5. STABILITY OF EQUATION (1)

We use these Lemmas in the proofs of Theorems 2 and 3, in which we put

$$D := \{(x,x), (-x,x), (x,-x), (-x,-x); x \in X^\star\}.$$

Theorem 2. *Let $g : X \to Y$ be a function satisfying the inequality*

$$\|g(2x+y) - 4g(x) - g(y) - g(x+y) + g(x-y)\| \leq \omega(x,y), \tag{16}$$

where $\omega : X^\star \times X^\star \to [0,\infty)$ is a function fulfilling the following conditions:

$$\begin{cases} \lim_{n\to\infty} \frac{1}{9^n}\omega(3^n x, 3^n y) = 0; \\ \sum_{n=0}^{\infty} \frac{1}{9^n}\omega(3^n u, 3^n v) < \infty \quad \textit{for all} \quad (u,v) \in D. \end{cases}$$

Then there exists a unique quadratic function $Q : X \to Y$ satisfying the estimate

$$\|Q(x) - g(x)\| \leq \sum_{n=0}^{\infty} \frac{1}{9^{n+1}}\varphi(3^n x, 3^n x) + \psi(x), \tag{17}$$

where

$$\varphi(x,y) = \frac{1}{2}[\omega(x,y) + \omega(-x,y) + \omega(x,-y) + \omega(-x,-y)] \tag{18}$$

and

$$\psi(x) = \frac{1}{6}[\frac{1}{2}[\omega(x,x) + \omega(-x,-x)] + \omega(x,-x) + \omega(-x,x) + \frac{1}{2}[\omega(x,-2x) + \omega(-x,2x)]]. \tag{19}$$

Proof. First observe that, because of (18) and the limit properties of the function ω stated in (16), the function φ satisfies (6). Let p and h be the even and the odd part, respectively, of the function g, i.e.,

$$p(x) = \frac{g(x) + g(-x)}{2}, \quad h(x) = \frac{g(x) - g(-x)}{2}, \; x \in X.$$

It is not hard to check that

$$\|p(2x+y) - 4p(x) - p(x+y) - p(y) + p(x-y)\| \leq \frac{1}{2}[\omega(x,y) + \omega(-x,-y)] \tag{20}$$

and

$$\|h(2x+y)-4h(x)-h(x+y)-h(y)+h(x-y)\| \le \frac{1}{2}[\omega(x,y)+\omega(-x,-y)]. \quad (21)$$

Setting $y=x$ and then $y=-x$ in inequality (20), we get

$$\|p(3x)-5p(x)-p(2x)+p(0)\| \le \frac{1}{2}[\omega(x,x)+\omega(-x,-x)]$$

and

$$\|-4p(x)-p(0)+p(2x)\| \le \frac{1}{2}[\omega(x,-x)+\omega(-x,x)].$$

Consequently,

$$\|p(3x)-9p(x)\| \le \frac{1}{2}[\omega(x,x)+\omega(-x,-x)+\omega(x,-x)+\omega(-x,x)].$$

Thus, because of (18),

$$\|\frac{1}{9}p(3x)-p(x)\| \le \frac{1}{9}\varphi(x,x). \quad (22)$$

It follows from inequality (22) that

$$\|\frac{1}{9^{n+1}}p(3^{n+1}x)-\frac{1}{9^n}p(3^nx)\| \le \frac{1}{9}\frac{1}{9^n}\varphi(3^nx,3^nx).$$

Taking $\lambda_n=9^{-n}, \beta_n=3^n, n\in\mathbb{N}$, from Lemmas 1 and 2 and Theorem 1, we infer that there exists a quadratic function $Q: X\to Y$ fulfilling the estimate

$$\|Q(x)-p(x)\| \le \sum_{n=0}^{\infty}\frac{1}{9^{n+1}}\varphi(3^nx,3^nx). \quad (23)$$

(Note that (20) is of form (5) with $r=5$.)
Now we are going to check the inequality

$$\|h(x)\| \le \psi(x). \quad (24)$$

Since h is odd, then $h(0)=0$. Setting $y=x$, $y=-x$ and finally $y=-2x$ in (21), we obtain

$$\begin{aligned}&\|h(3x)-5h(x)-h(2x)\| \le \tfrac{1}{2}[\omega(x,x)+\omega(-x,-x)],\\ &\|2h(2x)-4h(x)\| \le \omega(x,-x)+\omega(-x,x),\\ &\|h(3x)-3h(x)+h(2x)\| \le \tfrac{1}{2}[\omega(x,-2x)+\omega(-x,2x)].\end{aligned}$$

Consequently, by the triangle inequality

$$\begin{aligned}\|h(x)\| \le \frac{1}{6}[&\frac{\omega(x,x)+\omega(-x,-x)}{2}+\omega(x,-x)+\omega(-x,x)+\\ &+\frac{\omega(x,-2x)+\omega(-x,2x)}{2}].\end{aligned} \quad (25)$$

By virtue of (19) and (25), we obtain estimate (24). Because of (23), this is (17).

To prove the uniqueness of Q assume that $Q_1 : X \to Y$ is a quadratic function satisfying estimate (17). On account of a theorem proved in [7],

$$Q(3x) = 9Q(x) \quad \text{as well as} \quad Q_1(3x) = 9Q_1(x),\ x \in X.$$

Thus

$$\begin{aligned}\|Q(x) - Q_1(x)\| &= \frac{1}{9^k}\|Q(3^k x) - Q_1(3^k x)\| \leq \\ &\leq \frac{1}{9^k}\{\|Q(3^k x) - f(3^k x)\| + \|Q_1(3^k x) - f(3^k x)\|\} \leq \\ &\leq \frac{1}{9^k}\{2\sum_{n=0}^{\infty}\frac{1}{9^{n+1}}\varphi(3^{n+k}x, 3^{n+k}x)) + \psi(3^k x)\} = \\ &= 2\sum_{j=k}^{\infty}\frac{1}{9^{j+1}}\varphi(3^j x, 3^j x) + \frac{1}{9^k}\psi(3^k x).\end{aligned}$$

By our assumption, the last expression tends to zero, as $k \to \infty$. This completes the proof of Theorem 2. □

Theorem 3. *Assume that X is a commutative group uniquely divisible by 3. Let $g : X \to Y$ be a function satisfying the inequality*

$$\|g(2x + y) - 4g(x) - g(y) - g(x + y) + g(x - y)\| \leq \omega(x, y),$$

where $\omega : X^\star \times X^\star \to [0, \infty)$ is a function fulfilling the following conditions

$$\begin{cases}\lim_{n\to\infty} 9^n\omega(3^{-n}x, 3^{-n}y) = 0; \\ \sum_{n=0}^{\infty} 9^n\omega(3^{-n}u, 3^{-n}v) < \infty \quad \text{for all } (u, v) \in D.\end{cases}$$

Then there exists a unique quadratic function $Q : X \to Y$ satisfying the estimate

$$\|Q(x) - g(x)\| \leq \sum_{n=0}^{\infty} 9^n\varphi(3^{-n-1}x, 3^{-n-1}x) + \psi(x),$$

where φ and ψ are defined as in Theorem 2.

Proof. The proof runs similarly to the proof of Theorem 2. Let φ and ψ be defined as in Theorem 2. We consider inequalities (20) and (21). In the proof of Theorem 2, we obtained the following inequalities

$$\|h(x)\| \leq \psi(x)$$

and

$$\|p(3x) - 9p(x)\| \leq \varphi(x, x). \tag{26}$$

Replacing x by $\frac{x}{3}$ in (26), we get

$$\|9p(\frac{x}{3}) - p(x)\| \leq \varphi(\frac{x}{3}, \frac{x}{3})$$

whence

$$\|9^{n+1}p(\frac{x}{3^{n+1}}) - 9^n p(\frac{x}{3^n})\| \leq 9^n \varphi(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}).$$

It follows from the assumptions of Theorem 3 that

$$\begin{cases} \lim_{n\to\infty} 9^n \varphi(3^{-n-1}x, 3^{-n-1}y) = 0, \\ \sum_{n=0}^{\infty} 9^n \varphi(3^{-n-1}x, 3^{-n-1}x) < \infty. \end{cases}$$

On account of Lemmas 1 and 3, as well as Theorem 1, there exists a quadratic function $Q : X \to Y$ satisfying the following estimate

$$\|Q(x) - p(x)\| \leq \sum_{n=0}^{\infty} 9^n \varphi(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}).$$

Therefore,

$$\|Q(x) - g(x)\| \leq \sum_{n=0}^{\infty} 9^n \varphi(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}) + \psi(x).$$

The proof of the uniqueness of Q is quite similar to that of Theorem 2. □

6. STABILITY OF EQUATION (2)

Theorem 4. *Let $g : X \to Y$ be a function satisfying the inequality*

$$\|g(2x + y) - g(2x - y) - 8g(x) - 2g(y)\| \leq \varphi(x, y), \tag{27}$$

where $\varphi : X^\star \times X^\star \to [0, \infty)$ is a function fulfilling the conditions:

$$\begin{cases} \lim_{n\to\infty} \frac{1}{9^n}\varphi(3^n x, 3^n y) = 0; \\ \sum_{n=0}^{\infty} \frac{1}{9^n}\varphi(3^x x, 3^n x) \text{ is convergent.} \end{cases}$$

Then there exists a unique quadratic function $Q : X \to Y$ satisfying the following estimate

$$\|Q(x) - g(x)\| \leq \sum_{n=0}^{\infty} \frac{1}{9^{n+1}}\varphi(3^n x, 3^n x)). \tag{28}$$

Proof. Putting $y = x$ in (27), we get

$$\|g(3x) - 9g(x)\| \leq \varphi(x, x). \tag{29}$$

Now we argue quite similarly as in the proof of Theorem 2 (the even case), obtaining the existence of a unique quadratic function $Q : X \to Y$ fulfilling estimate (28). □

Theorem 5. *Assume that X is a commutative group uniquely divisible by 3. Let $g : X \to Y$ be a function satisfying the inequality*

$$\|g(2x+y) + g(2x-y) - 8g(x) - 2g(y)\| \le \varphi(x,y),$$

where $\varphi : X^\star \times X^\star \to [0,\infty)$ is a function fulfilling the following conditions:

$$\begin{cases} \lim_{n\to\infty} 9^n \varphi(3^{-n}x, 3^{-n}y) = 0; \\ \sum_{n=0}^{\infty} 9^n \varphi(3^{-n}x, 3^{-n}x) < \infty. \end{cases}$$

Then there exists a unique quadratic function $Q : X \to Y$ satisfying the estimate

$$\|Q(x) - g(x)\| \le \sum_{n=0}^{\infty} 9^n \varphi(3^{-n-1}x, 3^{-n-1}x).$$

Proof. Setting $\frac{x}{3}$ instead of x in (29), we obtain

$$\|g(x) - 9g(\frac{x}{3})\| \le \varphi(\frac{x}{3}, \frac{x}{3}).$$

Now we argue as in the proof of Theorem 3, obtaining a unique quadratic function Q fulfilling the following estimate

$$\|Q(x) - g(x)\| \le \sum_{n=0}^{\infty} 9^n \varphi(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}). \qquad \square$$

Concluding remark. *Let $\theta \ge 0$ and $\omega(x,y) = \theta(\|x\|^\alpha + \|y\|^\alpha)$, or $\omega(x,y) = \theta\|x\|^\beta\|y\|^\beta$. Theorems 2 and 4 can be applied to these functions ω with $\alpha < 2$ and $\beta < 1$, whereas Theorems 3 and 5 – with $\alpha > 2$ and $\beta > 1$. Thus our theorems cover the cases considered by several other authors, and in particular, by C. Park and J. Su An.*

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Zygfryd Kominek
zkominek@ux2.math.us.edu.pl

Silesian University
Institute of Mathematics
ul. Bankowa 14, 40-007 Katowice, Poland

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