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Author: Jacek Chmieliński, Justyna Sikorska, Paweł Wójcik

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On a ρ -Orthogonally Additive Mappings

Jacek Chmieliński, Justyna Sikorska, and Paweł Wójcik

Abstract. We show that a real normed linear space endowed with the ρ -orthogonality relation, in general need not be an orthogonality space in the sense of Rätz. However, we prove that ρ -orthogonally additive mappings defined on some classical Banach spaces have to be additive. Moreover, additivity (and approximate additivity) under the condition of an approximate orthogonality is considered.

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1. Introduction

Inner product spaces are by all means the most natural venue for orthogonality, allowing the definition: $x \perp y \Leftrightarrow \langle x|y \rangle = 0$. However, analogous relations may be considered also in normed linear spaces, as well as in more general settings.

In a real normed linear space $(X, \|\cdot\|)$, for two vectors $x, y \in X$, one can consider for example the *Birkhoff-James orthogonality* \perp_B (see [2, 15]) defined by

$$x \perp_B y \iff \forall \lambda \in \mathbb{R} \quad \|x\| \leq \|x + \lambda y\|,$$

or the *isosceles orthogonality* \perp_i (see [15]) defined by

$$x \perp_i y \iff \|x + y\| = \|x - y\|,$$

or many others. Moreover, some axiomatic definitions of the orthogonality in linear spaces (or even more general structures) are known, among them the one formulated by Rätz [18] (compare with [12]). A real linear space X of dimension

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at least 2 with a binary relation $\perp \subseteq X \times X$ is called an *orthogonality space* (in the sense of Rätz) whenever the following conditions are satisfied:

- (ort1) $x \perp 0$ and $0 \perp x$ for all $x \in X$;
- (ort2) if $x, y \in X \setminus \{0\}$ and $x \perp y$, then x and y are linearly independent;
- (ort3) if $x, y \in X$ and $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
- (ort4) for any two-dimensional subspace P of X , for every $x \in P$ and for every $\lambda \in [0, +\infty)$ there exists $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

It is easily seen that any inner product space is an orthogonality space in the above sense. Not so easily, but still the same can be shown for an arbitrary real normed linear space with the Birkhoff-James orthogonality \perp_B (see [18]).

Yet another axiomatic definition of orthogonality, considered merely on groups, was introduced by Fechner and Sikorska [10]; we refer to it in Sect. 4.

Let X be a suitable algebraic structure with an orthogonality relation \perp (in whatsoever sense) and let G be a group (usually abelian). The following conditional functional equation, with $f: X \rightarrow G$, will be the key notion of the paper:

$$x \perp y \implies f(x + y) = f(x) + f(y), \quad x, y \in X. \tag{1.1}$$

The function f is then called an *orthogonally additive mapping* and the history of that concept goes back nearly a century. We refer to surveys by Rätz [19] and more recent by Sikorska [20] for motivations, history, various aspects and problems connected with the subject.

In the basic case of X being a real inner product space it is known (cf. [5, 18, 26]) that any orthogonally additive mapping f must be of the form

$$f(x) = a(\|x\|^2) + b(x), \quad x \in X, \tag{1.2}$$

with unique additive mappings $a: \mathbb{R} \rightarrow G$ and $b: X \rightarrow G$. This yields, in particular, that an orthogonally additive mapping defined on an inner product space need not be additive. Notice that commutativity of the group G need not be assumed here (see the recent proof by Toborg [26, Theorem 3.3]). In more general cases, orthogonally additive mappings have been also widely investigated (see e.g., [6, 12, 18]). We recall here the theorem of Rätz, Baron and Volkman concerning the orthogonal additivity for mappings defined on an orthogonality space.

Theorem 1.1. ([6, 18]) *Let (X, \perp) be an orthogonality space in the sense of Rätz and let G be an abelian group. A mapping $f: X \rightarrow G$ satisfies the conditional equation*

$$x \perp y \implies f(x + y) = f(x) + f(y), \quad x, y \in X$$

if and only if there exist an additive mapping $\varphi: X \rightarrow G$ and a biadditive and symmetric mapping $\Phi: X \times X \rightarrow G$ such that

$$f(x) = \varphi(x) + \Phi(x, x) \quad x \in X \tag{1.3}$$

and $\Phi(x, y) = 0$ for all $x, y \in X$ such that $x \perp y$.

It follows that all *Birkhoff-James orthogonally additive mappings* (\perp_B -additive mappings), i.e., such that $x \perp_B y$ implies $f(x + y) = f(x) + f(y)$, have the form (1.3).

Quite surprisingly, it turns out that in the setting of the orthogonality space (X, \perp) , all orthogonally additive mappings are unconditionally additive, unless X is an inner product space, i.e., unless there exists an inner product $\langle \cdot | \cdot \rangle$ in X such that $x \perp y$ if and only if $\langle x | y \rangle = 0$. Equivalently, only inner product spaces admit nonzero and even orthogonally additive mappings. This was proved by Szabó [23] when $\dim X \geq 3$ and by Yang [27] for the remaining case $\dim X = 2$. Therefore, all \perp_B -additive mappings are additive, unless the norm $\|\cdot\|$ comes from an inner product. Actually, for this particular orthogonality, the phenomenon was observed earlier —see [16, 21, 22]. The above property, however, is not restricted to orthogonality spaces. In particular, in a normed linear space of the dimension greater than 2 and with the isosceles orthogonality \perp_i , the existence of orthogonally additive mappings which are not additive also characterizes inner product spaces —see [24, 25].

Let us go back to the case of an inner product space X . For a nonnegative constant ε , a notion of ε -orthogonality (*approximate orthogonality*) of vectors $x, y \in X$ can be introduced in a natural way by

$$x \perp^\varepsilon y \iff |\langle x | y \rangle| \leq \varepsilon \|x\| \|y\|.$$

Surely, $\perp^0 = \perp$ and since for $\varepsilon \geq 1$ the Cauchy-Schwarz inequality leads to $\perp^\varepsilon = X^2$, we restrict the range of ε to the interval $[0, 1)$.

Having defined an approximate orthogonality on X , we may consider, for a group G and a mapping $f: X \rightarrow G$, a stronger than (1.1) condition

$$x \perp^\varepsilon y \implies f(x + y) = f(x) + f(y), \quad x, y \in X. \tag{1.4}$$

Obviously, for $\varepsilon = 0$, (1.4) coincides with (1.1).

Although an orthogonally additive mapping defined on an inner product space need not be additive, any solution of (1.4) with a strictly positive ε is unconditionally additive.

Theorem 1.2. *Let X be a real inner product space with $\dim X \geq 2$, let G be a group and let $\varepsilon \in (0, 1)$. A function $f: X \rightarrow G$ satisfies (1.4) if and only if f is additive.*

Let us stress that we do not assume commutativity of G here, although, as in the rest of the paper, we use an additive notation.

Proof. It is clear that additivity of f implies (1.4). For the reverse, assume (1.4) whence f must be of the form (1.2). In order to prove additivity of f it is enough to show that the mapping a vanishes. Of course $a(0) = 0$, so we fix an arbitrary $t \in \mathbb{R} \setminus \{0\}$ and by proving that $a(t) = 0$ we will finish the proof. Let us choose two unit vectors u, v such that $\langle u | v \rangle = \varepsilon$. Then for $x := \frac{t}{2\varepsilon} u$

and $y := v$ we have

$$\|x\| = \frac{|t|}{2\varepsilon}, \quad \|y\| = 1, \quad t = 2 \langle x|y \rangle.$$

Therefore $\frac{\langle x|y \rangle}{\|x\| \|y\|} = \varepsilon$, i.e., $x \perp^\varepsilon y$ as well as $x \perp^\varepsilon (-y)$, whence $f(x + y) = f(x) + f(y)$ and $f(x - y) = f(x) + f(-y)$. Then (1.2) yields

$$\begin{aligned} a(\|x + y\|^2) + b(x) &= a(\|x\|^2) + b(x) + a(\|y\|^2), \\ a(\|x - y\|^2) + b(x) &= a(\|x\|^2) + b(x) + a(\|y\|^2); \end{aligned}$$

consequently $a(\|x + y\|^2) = a(\|x - y\|^2)$ and $a(t) = a(2 \langle x|y \rangle) = 0$ follows. \square

There are examples (cf. [3]) of injective or surjective (but not bijective —cf. [4]) orthogonally additive mappings which are not additive. Theorem 1.2 shows that none of them may satisfy (1.4) with any positive $\varepsilon \in (0, 1)$.

The above considerations and results have motivated our work, which is devoted to an orthogonality relation \perp_ρ defined in a real normed linear space and to the corresponding with it orthogonal additivity.

The paper is organized as follows. In Sect. 2, using the notion of *norm derivatives*, we introduce the mapping ρ' and define the ρ -orthogonality. In the subsequent part we consider ρ -orthogonally additive mappings defined on some particular real normed linear spaces. In Sect. 4 we extend Theorem 1.2 to normed linear spaces and we investigate mappings which are approximately additive under the condition of an approximate orthogonality. We finish our paper with some concluding remarks and open problems.

2. Norm Derivatives and ρ -Orthogonality

In this section we define and consider yet another orthogonality relation in a real normed linear space $(X, \|\cdot\|)$. First, we recall the notion of the so-called (right and left) *norm derivatives* $\rho'_+, \rho'_- : X \times X \rightarrow \mathbb{R}$ (see e.g., [1, 7, 9]):

$$\rho'_\pm(x, y) := \lim_{\lambda \rightarrow 0^\pm} \frac{\|x + \lambda y\|^2 - \|x\|^2}{2\lambda} = \|x\| \cdot \lim_{\lambda \rightarrow 0^\pm} \frac{\|x + \lambda y\| - \|x\|}{\lambda}, \quad x, y \in X.$$

Convexity of the norm yields that the limits exist and the above definitions are meaningful. It is also natural to consider (cf. [17]) the mapping $\rho' : X \times X \rightarrow \mathbb{R}$ being the arithmetic mean of ρ'_+ and ρ'_- , i.e.,

$$\rho'(x, y) := \frac{1}{2} (\rho'_-(x, y) + \rho'_+(x, y)), \quad x, y \in X.$$

The following properties of ρ' will be useful (for the proofs consult, e.g., [1, 9, 17]):

- (m1) $\rho'(x, \alpha x + y) = \alpha \|x\|^2 + \rho'(x, y)$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$;
- (m2) $\rho'(\alpha x, \beta y) = \alpha \beta \rho'(x, y)$ for all $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$;
- (m3) $\rho'(x, x) = \|x\|^2$ and $|\rho'(x, y)| \leq \|x\| \|y\|$ for all $x, y \in X$;
- (m4) $\rho'(x, \cdot)$ is continuous for every $x \in X$.

Note, that ρ' need not be continuous with respect to the first variable.

The mapping ρ' can thus be regarded as a substitute of an inner product in X (and so can be ρ'_+ and ρ'_-). The following definition is therefore natural and it plays a crucial role in the present paper. We define a ρ -orthogonality relation, denoted by \perp_ρ , as

$$x \perp_\rho y \iff \rho'(x, y) = 0, \quad x, y \in X.$$

It is easy to check that the relation \perp_ρ satisfies the Rätz axioms (ort1), (ort2) and (ort3). However, as shown in the example below, the axiom (ort4) need not be fulfilled in general. We are aware that this contradicts the statement of [1, Proposition 2.8.1], the proof of which is unfortunately incorrect.

Example 2.1. Consider the space $l_\infty^2 := (\mathbb{R}^2, \|\cdot\|_\infty)$ with $\|(x_1, x_2)\|_\infty := \max\{|x_1|, |x_2|\}$ for $(x_1, x_2) \in \mathbb{R}^2$. In this particular space, explicit formulas for ρ'_+ and ρ'_- can be obtained (see [1, Example 2.1.2]). Namely, for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ we have

$$\begin{aligned} \rho'_+((x_1, x_2), (y_1, y_2)) &= \max\{x_k y_k : k = 1, 2, |x_k| = \|(x_1, x_2)\|_\infty\}, \\ \rho'_-((x_1, x_2), (y_1, y_2)) &= \min\{x_k y_k : k = 1, 2, |x_k| = \|(x_1, x_2)\|_\infty\} \end{aligned}$$

and consequently

$$\rho'((x_1, x_2), (y_1, y_2)) = \begin{cases} x_1 y_1 & \text{whenever } |x_1| > |x_2|, \\ x_2 y_2 & \text{whenever } |x_1| < |x_2|, \\ \frac{x_1 y_1 + x_2 y_2}{2} & \text{whenever } |x_1| = |x_2|. \end{cases} \quad (2.1)$$

Consider the vector $(1, \frac{1}{4}) \in \mathbb{R}^2$; it follows from (2.1) that for any $(\alpha, \beta) \in \mathbb{R}^2$ there is

$$(1, \frac{1}{4}) \perp_\rho (\alpha, \beta) \iff \alpha = 0.$$

If (ort4) were true we would have, for some $\beta \in \mathbb{R}$,

$$(1, \frac{1}{4}) + (0, \beta) \perp_\rho (1, \frac{1}{4}) - (0, \beta),$$

and hence

$$\rho'((1, \frac{1}{4} + \beta), (1, \frac{1}{4} - \beta)) = 0.$$

It follows from (2.1) that the last equality is impossible; indeed, if $|\frac{1}{4} + \beta| < 1$, then

$$\rho'((1, \frac{1}{4} + \beta), (1, \frac{1}{4} - \beta)) = 1 \neq 0;$$

if $|\frac{1}{4} + \beta| > 1$, then

$$\rho'((1, \frac{1}{4} + \beta), (1, \frac{1}{4} - \beta)) = \frac{1}{16} - \beta^2 \neq 0$$

(if $\frac{1}{16} - \beta^2 = 0$ were true, then $|\frac{1}{4} + \beta| > 1$ would not be satisfied); and finally if $|\frac{1}{4} + \beta| = 1$, then $\beta = \frac{3}{4}$ or $\beta = -\frac{5}{4}$ and $\rho'((1, \frac{1}{4} + \beta), (1, \frac{1}{4} - \beta)) = \pm \frac{1}{4} \neq 0$.

The space l_∞^2 is not exceptional in lacking the (ort4) condition; on the contrary, the family of such spaces is quite large. Let $(X_1, \|\cdot\|_1), (X_2, \|\cdot\|_2)$ be nontrivial real normed linear spaces. By $X_1 \oplus_\infty X_2$ we denote the product space $X_1 \times X_2$ with the norm $\|(x_1, x_2)\|_\infty := \max\{\|x_1\|_1, \|x_2\|_2\}$. Now, the space $X_1 \oplus_\infty X_2$ contains a subspace isometric to l_∞^2 and therefore, as a consequence of what we observed in Example 2.1, it cannot satisfy (ort4).

Theorem 2.2. *The relation \perp_ρ in $X_1 \oplus_\infty X_2$ does not satisfy (ort4).*

Notice that some classical spaces like l_∞, c, c_o have the form $X_1 \oplus_\infty X_2$ (for example c_o is isometric to $\mathbb{R} \oplus_\infty c_o$). The fact that the space (X, \perp_ρ) need not be an orthogonality space in the sense of Rätz, motivates our investigations in the next section.

3. ρ -Orthogonal Additivity

We aim at a characterization of ρ -orthogonally additive mappings, i.e., those satisfying

$$x \perp_\rho y \implies f(x + y) = f(x) + f(y) \tag{3.1}$$

in spaces for which (in view of Theorem 2.2) we cannot apply Theorem 1.1.

In the first part of this section, we consider the case where the domain is a product of two real normed linear spaces and the relation \perp_ρ corresponds to the relevant norm $\|\cdot\|_\infty$ in that space. We start with the following lemma.

Lemma 3.1. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be real normed linear spaces that yield the product space $X_1 \oplus_\infty X_2$ and the orthogonality relation \perp_ρ in it. For arbitrary $x_1 \in X_1$ and $x_2 \in X_2$ the following statements are true.*

- (i) *If $\|x_1\|_1 > \|x_2\|_2$, then $(x_1, x_2) \perp_\rho (0, x'_2)$ for all $x'_2 \in X_2$.*
- (ii) *If $\|x_1\|_1 < \|x_2\|_2$, then $(x_1, x_2) \perp_\rho (x'_1, 0)$ for all $x'_1 \in X_1$.*

Proof. We will prove (i) only, and the proof of (ii) runs similarly. We fix $x'_2 \in X_2$ and observe that

$$\begin{aligned} \rho'_+((x_1, x_2), (0, x'_2)) &= \|(x_1, x_2)\|_\infty \cdot \lim_{t \rightarrow 0^+} \frac{\|(x_1, x_2) + t(0, x'_2)\|_\infty - \|(x_1, x_2)\|_\infty}{t} \\ &= \|(x_1, x_2)\|_\infty \\ &\quad \cdot \lim_{t \rightarrow 0^+} \frac{\max\{\|x_1\|_1, \|x_2 + tx'_2\|_2\} - \max\{\|x_1\|_1, \|x_2\|_2\}}{t}. \end{aligned}$$

Since $\|x_1\|_1 > \|x_2\|_2$, there exists $t_1 > 0$ such that $\|x_1\|_1 > \|x_2 + tx'_2\|_2$ for all $t \in (0, t_1)$. Thus we have

$$\begin{aligned} &\lim_{t \rightarrow 0^+} \frac{\max\{\|x_1\|_1, \|x_2 + tx'_2\|_2\} - \max\{\|x_1\|_1, \|x_2\|_2\}}{t} \\ &= \lim_{t \rightarrow 0^+} \frac{\|x_1\|_1 - \|x_1\|_1}{t} = 0 \end{aligned}$$

and it follows that $\rho'_+((x_1, x_2), (0, x'_2)) = 0$. Similarly we prove that $\rho'_-((x_1, x_2), (0, x'_2)) = 0$, whence $\rho'((x_1, x_2), (0, x'_2)) = 0$ and finally, $(x_1, x_2) \perp_\rho (0, x'_2)$. \square

In the following theorem, given two normed linear spaces, we characterize ρ -orthogonally additive mappings defined on the product of these spaces and taking values in an arbitrary group. We stress here that the commutativity of the target group is not assumed.

Theorem 3.2. *Let G be a group, let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be nontrivial real normed linear spaces and let $f: X_1 \oplus_\infty X_2 \rightarrow G$ satisfy*

$$x \perp_\rho y \implies f(x + y) = f(x) + f(y), \quad x, y \in X_1 \times X_2. \tag{3.2}$$

Then f is additive.

Proof. First we prove that f is additive on $\{0\} \times X_2$. Fix $(0, x'_2), (0, x''_2) \in \{0\} \times X_2$ and take any $x_1 \in X_1$ such that $\|x_1\|_1 > \|x'_2\|_2$. By Lemma 3.1(i), $(x_1, 0) \perp_\rho (0, x'_2)$, $(x_1, x'_2) \perp_\rho (0, x''_2)$ and $(x_1, 0) \perp_\rho (0, x'_2 + x''_2)$. Applying (3.2) we get

$$\begin{aligned} f(x_1, 0) + f(0, x'_2) + f(0, x''_2) &= f((x_1, 0) + (0, x'_2)) + f(0, x''_2) \\ &= f(x_1, x'_2) + f(0, x''_2) = f((x_1, x'_2) + (0, x''_2)) \\ &= f((x_1, 0) + (0, x'_2 + x''_2)) \\ &= f(x_1, 0) + f(0, x'_2 + x''_2) \end{aligned}$$

whence $f(0, x'_2) + f(0, x''_2) = f((0, x'_2) + (0, x''_2))$ and additivity of $f|_{\{0\} \times X_2}$ is proved. In a similar way (using Lemma 3.1(ii)), one checks that $f|_{X_1 \times \{0\}}$ is additive.

To prove additivity of f on the whole space, fix $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$. Lemma 3.1 and the fact that \perp_ρ satisfies (ort1) yield $X_1 \times \{0\} \perp_\rho \{0\} \times X_2$ and $\{0\} \times X_2 \perp_\rho X_1 \times \{0\}$. Thus we get from (3.2) and from additivity of $f|_{\{0\} \times X_2}$ and $f|_{X_1 \times \{0\}}$

$$\begin{aligned} f(x + y) &= f((x_1 + y_1, 0) + (0, x_2 + y_2)) = f(x_1 + y_1, 0) + f(0, x_2 + y_2) \\ &= f(x_1, 0) + f(y_1, 0) + f(0, x_2) + f(0, y_2) \\ &= f(x_1, 0) + f((y_1, 0) + (0, x_2)) + f(0, y_2) \\ &= f(x_1, 0) + f((0, x_2) + (y_1, 0)) + f(0, y_2) \\ &= f(x_1, 0) + f(0, x_2) + f(y_1, 0) + f(0, y_2) \\ &= f((x_1, 0) + (0, x_2)) + f((y_1, 0) + (0, y_2)) = f(x) + f(y). \end{aligned}$$

Thus f is additive. \square

Since $\perp_\rho \subseteq \perp_B$ (see e.g., [7, 8]), from Theorem 3.2 we get immediately a novel, in a sense, property of Birkhoff-James orthogonally additive mappings.

Theorem 3.3. *Let G be a group, let X_1 and X_2 be nontrivial real normed linear spaces and let $f: X_1 \oplus_\infty X_2 \rightarrow G$ satisfy*

$$x \perp_{\text{By}} y \implies f(x + y) = f(x) + f(y), \quad x, y \in X_1 \times X_2.$$

Then f is additive.

We emphasize that commutativity of G is not assumed, so the above result cannot be derived from the already known ones. In particular, it does not follow from Theorem 1.1 or from the results of Szabó [23] and Yang [27], even though $(X_1 \oplus_\infty X_2, \perp_{\text{B}})$ is an orthogonality space.

Let us consider now a classical Banach space $\mathcal{C}[0, 1]$ of continuous real-valued functions on $[0, 1]$, with the supremum norm. For mappings $\varphi, \psi \in \mathcal{C}[0, 1]$ defined by

$$\varphi(t) = t, \quad \psi(t) = 1 - t, \quad t \in [0, 1]$$

the subspace $\text{span}\{\varphi, \psi\} \subseteq \mathcal{C}[0, 1]$ is isometric to l_∞^2 . Thus the orthogonality \perp_ρ in $\mathcal{C}[0, 1]$ does not satisfy (ort4). Nevertheless, we are able to characterize ρ -orthogonally additive mappings defined on that space. In the first result we make no commutativity assumption for the target group.

Theorem 3.4. *Let G be a group and let $f: \mathcal{C}[0, 1] \rightarrow G$ satisfy*

$$\varphi \perp_\rho \psi \implies f(\varphi + \psi) = f(\varphi) + f(\psi), \quad \varphi, \psi \in \mathcal{C}[0, 1]. \tag{3.3}$$

Then there exist additive mappings $a, b: \mathcal{C}[0, 1] \rightarrow G$ such that $f = a + b$.

Proof. For a fixed number $t_1 \in (0, 1)$ let $M_{t_1}, N_{t_1} \subseteq \mathcal{C}[0, 1]$ be the subspaces defined by

$$M_{t_1} := \{\varphi \in \mathcal{C}[0, 1] : \varphi|_{[t_1, 1]} = 0\} \quad \text{and} \quad N_{t_1} := \{\varphi \in \mathcal{C}[0, 1] : \varphi|_{[0, t_1]} = 0\}.$$

Next, define a continuous linear functional $\eta_{t_1}: \mathcal{C}[0, 1] \rightarrow \mathbb{R}$ by $\eta_{t_1}(\varphi) := \varphi(t_1)$. It is easy to check that $\ker \eta_{t_1} = M_{t_1} \oplus N_{t_1}$. Moreover, it is not difficult to show that the subspace $M_{t_1} \oplus N_{t_1} \subseteq \mathcal{C}[0, 1]$ is isometrically isomorphic to the Banach space $M_{t_1} \oplus_\infty N_{t_1}$. Thus, we may identify $M_{t_1} \oplus_\infty N_{t_1}$ with $\ker \eta_{t_1}$ and we may consider $M_{t_1} \oplus_\infty N_{t_1}$ as a closed subspace of $\mathcal{C}[0, 1]$ with $\text{codim } M_{t_1} \oplus_\infty N_{t_1} = 1$. From Theorem 3.2 we know that $f|_{\ker \eta_{t_1}}$ is additive.

Now, fix a number $t_2 \in (0, 1) \setminus \{t_1\}$ and define $\eta_{t_2}: \mathcal{C}[0, 1] \rightarrow \mathbb{R}$ by $\eta_{t_2}(\varphi) := \varphi(t_2)$. Similarly as before one proves that $f|_{\ker \eta_{t_2}}$ is additive.

We choose an arbitrary function $\varphi_1 \in \mathcal{C}[0, 1]$ such that $\|\varphi_1\| = 1$ and

$$0 = \varphi_1(t_2) < \varphi_1(t) < \varphi_1(t_1) = 1, \quad t \in [0, 1] \setminus \{t_1, t_2\}.$$

In particular, we have $\varphi_1 \notin \ker \eta_{t_1}$ and since $\text{codim } \ker \eta_{t_1} = 1$, we have

$$\mathcal{C}[0, 1] = \text{span}\{\varphi_1\} + \ker \eta_{t_1}.$$

Using explicit formulas for ρ'_+ and ρ'_- in $\mathcal{C}[0, 1]$ given in [1, Example 2.1.5]

$$\begin{aligned} \rho'_+(\varphi, \psi) &= \sup\{\varphi(t)\psi(t) : t \in [0, 1], |\varphi(t)| = \|\varphi\|\}, \\ \rho'_-(\varphi, \psi) &= \inf\{\varphi(t)\psi(t) : t \in [0, 1], |\varphi(t)| = \|\varphi\|\}, \end{aligned}$$

we arrive at

$$\rho'_+(\varphi_1, \gamma) = \rho'_-(\varphi_1, \gamma) = \varphi_1(t_1)\gamma(t_1) = \gamma(t_1) \quad \gamma \in \mathcal{C}[0, 1].$$

Consequently, $\rho'(\varphi_1, \gamma) = \gamma(t_1)$ and thus, in particular,

$$\varphi_1 \perp_\rho \ker \eta_{t_1}. \tag{3.4}$$

For a vector in $\mathcal{C}[0, 1]$ with a unique decomposition $\alpha\varphi_1 + h \in \text{span}\{\varphi_1\} + \ker \eta_{t_1} = \mathcal{C}[0, 1]$ we define a linear projection $P: \mathcal{C}[0, 1] \rightarrow \text{span}\{\varphi_1\}$ by the formula $P(\alpha\varphi_1 + h) := \alpha\varphi_1$. It follows that $\ker P = \ker \eta_{t_1}$ and $P \circ P = P$. Therefore, we obtain $\psi - P(\psi) \in \ker \eta_{t_1}$ for all $\psi \in \mathcal{C}[0, 1]$, and thus, by (3.4), $P(\psi) \perp_\rho \psi - P(\psi)$.

Let $a, b: \mathcal{C}[0, 1] \rightarrow G$ be defined by $a(\psi) := f(P(\psi))$ and $b(\psi) := f(\psi - P(\psi))$ for $\psi \in \mathcal{C}[0, 1]$. We know that $f|_{\ker \eta_{t_1}}$ and $f|_{\ker \eta_{t_2}}$ are additive. Since $\text{span}\{\varphi_1\} \subseteq \ker \eta_{t_2}$, f is additive on $\text{span}\{\varphi_1\}$. Therefore, both mappings a and b are additive. Finally, we have from (3.3),

$$f(\psi) = f(P(\psi) + \psi - P(\psi)) = f(P(\psi)) + f(\psi - P(\psi)) = a(\psi) + b(\psi),$$

and the proof is complete. □

Assuming that G is abelian, the sum of mappings a and b is additive, whence our result takes the following form.

Theorem 3.5. *Let G be an abelian group and let $f: \mathcal{C}[0, 1] \rightarrow G$ be a ρ -orthogonally additive mapping (i.e., satisfy (3.3)). Then f is additive.*

A natural problem arises.

Problem 3.6. Is the commutativity of G a necessary assumption in Theorem 3.5?

In order to answer this question positively, it suffices to prove that if two mappings $a, b: \mathcal{C}[0, 1] \rightarrow G$ are additive and their sum $f := a + b$ satisfies (3.3), then f is additive. We can relate this problem with the result of Toborg [26, Corollary 3.4] who proved that in the case of an inner product space X in the domain, the subgroup of G generated by the image $f(X)$ of an orthogonally additive mapping f is abelian.

Another question is whether the results obtained for $X_1 \oplus_\infty X_2$ and $\mathcal{C}[0, 1]$ can be extended to other, or perhaps all, spaces. The exact statement of this problem concludes the section.

Problem 3.7. Is it true that in any normed linear space X which is not an inner product space, any ρ -orthogonally additive mapping is additive?

4. Approximate ρ -Orthogonal Additivity and Stability

In the final part of the paper we deal with *approximate* solutions of equation (3.1) as well as with its stability.

We start with a simple general observation. Let $(F, +)$ be a semigroup, let $(G, +)$ be an abelian group, and let $\Delta \subseteq F \times F$ and $U \subseteq G$ be nonempty subsets. Suppose that mappings $f, g: F \rightarrow G$ satisfy

$$f(x) - g(x) \in U, \quad x \in F$$

and g is additive on Δ , i.e.,

$$g(x + y) = g(x) + g(y), \quad (x, y) \in \Delta. \tag{4.1}$$

Then for a pair $(x, y) \in \Delta$ we have $-g(x + y) + g(x) + g(y) = 0$ and also

$$f(x + y) - g(x + y) \in U, \quad -f(x) + g(x) \in -U, \quad -f(y) + g(y) \in -U.$$

Adding it all (with the help of commutativity), we obtain

$$f(x + y) - f(x) - f(y) \in U + (-U) + (-U), \quad (x, y) \in \Delta. \tag{4.2}$$

Example 4.1. Keeping the above notation, suppose that a mapping $g: F \rightarrow G$ satisfies the property (4.1). For each element $x \in F$, using the axiom of choice, we take an arbitrary $a_x \in g(x) + U$ and define a function $f: F \rightarrow G$ by $f(x) := a_x$. Then $f(x) - g(x) \in U$ for each $x \in F$ and it follows that f satisfies (4.2). This shows that the family of mappings satisfying (4.2) can be large and these mappings may be very irregular.

Assume now that for a binary relation \perp in F we have $x \perp y \Leftrightarrow (x, y) \in \Delta$ and assume that the target group G is replaced by a normed linear space $(Y, \|\cdot\|)$ with U being a closed ball centred at zero, with the radius $\delta \geq 0$. Suppose that a mapping $g: F \rightarrow Y$ satisfies (4.1), which now takes the form

$$x \perp y \implies g(x + y) = g(x) + g(y), \quad x, y \in F.$$

If a mapping $f: F \rightarrow Y$ satisfies $\|f(x) - g(x)\| \leq \delta$ for all $x \in F$, then f satisfies (4.2), the meaning of which is

$$x \perp y \implies \|f(x + y) - f(x) - f(y)\| \leq 3\delta, \quad x, y \in F.$$

The last property will be of our interest in the present section with the role of \perp played by the ρ -orthogonality or an approximate ρ -orthogonality. We would like to ask a reverse, in a sense, question whether an approximately orthogonally additive mapping can be approximated by an orthogonally additive (or additive) one. The first result in this direction was obtained by Ger and Sikorska in [11] and then improved by Fechner and Sikorska in [10]. In the latter paper the authors considered an abelian group A with a binary relation \perp on it, with the properties:

- (ort-a) if $x, y \in A$ and $x \perp y$, then $x \perp -y$, $-x \perp y$ and $2x \perp 2y$;
- (ort-b) for every $x \in A$ there exists $y \in A$ such that $x \perp y$ and $x + y \perp x - y$.

With such settings, (A, \perp) is a generalization of an orthogonality space in the sense of Rätz.

It can be derived from the main theorem of [10] that for A being a uniquely 2-divisible abelian group, Y being a Banach space and $f: A \rightarrow Y$ satisfying, with $\delta \geq 0$, the condition

$$x \perp y \implies \|f(x+y) - f(x) - f(y)\| \leq \delta, \quad x, y \in A,$$

there exists a mapping $g: A \rightarrow Y$ such that

$$x \perp y \implies g(x+y) = g(x) + g(y), \quad x, y \in A$$

and $\|f(z) - g(z)\| \leq 5\delta$ for all $z \in A$.

For a pair of vectors x, y in a real normed linear space X we define their *approximate ρ -orthogonality*, or to be more precise, *ε - ρ -orthogonality*, with $\varepsilon \in [0, 1)$, by

$$x \perp_{\rho}^{\varepsilon} y \iff |\rho'(x, y)| \leq \varepsilon \|x\| \|y\|$$

(see [7, 8]). If the norm comes from an inner product, then $\perp_{\rho}^{\varepsilon}$ coincides with \perp^{ε} defined in the introduction.

As we have actually observed in Example 2.1, the relation \perp_{ρ} does not satisfy (ort-b) in l_{∞}^2 . It can be shown that the relation $\perp_{\rho}^{\varepsilon}$, for $\varepsilon < \frac{1}{4}$, does not satisfy (ort-b) in that space as well (we omit, however, a tedious verification of this fact).

Now, we are ready to prove the main results of this section.

Theorem 4.2. *Let X be a real normed linear space with $\dim X \geq 2$, let G be an abelian group and let D be a nonempty subset of G . Suppose that a function $f: X \rightarrow G$ satisfies, with $\varepsilon \in (0, 1)$, the condition*

$$x \perp_{\rho}^{\varepsilon} y \implies f(x+y) - f(x) - f(y) \in D, \quad x, y \in X. \tag{4.3}$$

Then

$$f(x+y) - f(x) - f(y) \in \tilde{D}, \quad x, y \in X, \tag{4.4}$$

where $\tilde{D} := D + (-D) + D + D + (-D)$.

We will call a mapping f satisfying (4.4), \tilde{D} -additive.

Proof. Fix a unit vector $u \in X$. Since $\rho'(u, \cdot)$ is continuous, there exists another unit vector $v \in X$ such that $\rho'(u, v) = \varepsilon$, whence $u \perp_{\rho}^{\varepsilon} v$. Fix $\alpha, \beta \in \mathbb{R}$ and let $y := -\frac{\beta}{\varepsilon}v$. Thus $y \in \text{span}\{v\}$ and $\rho'(u, y) = -\beta$. From the properties (m1) and (m2) we get

$$\alpha u \perp_{\rho} \beta u + y, \quad \alpha u + \beta u \perp_{\rho}^{\varepsilon} y, \quad \beta u \perp_{\rho}^{\varepsilon} y,$$

and it follows from (4.3) that

$$\begin{aligned} f(\alpha u + \beta u + y) - f(\alpha u + \beta u) - f(y) &\in D, \\ f(\alpha u + \beta u + y) - f(\alpha u) - f(\beta u + y) &\in D, \\ f(\beta u + y) - f(\beta u) - f(y) &\in D. \end{aligned}$$

Combining the last three statements, we get

$$f(\alpha u + \beta u) - f(\alpha u) - f(\beta u) \in (-D) + D + D, \quad \alpha, \beta \in \mathbb{R},$$

which means that the function $f|_{\text{span}\{u\}} : \text{span}\{u\} \rightarrow G$ is $((-D) + D + D)$ -additive.

Since our goal is to prove (4.4), we fix $x, y \in X \setminus \{0\}$ (if $x = 0$ or $y = 0$, the assertion follows easily since $D \subseteq \tilde{D}$). Let U be a two-dimensional subspace of X with $\text{span}\{x, y\} \subseteq U$. It follows from (m1) that there exists $z \in U \setminus \{0\}$ such that $x \perp_\rho z$ and it is easy to check that x and z are linearly independent. Therefore, $\text{span}\{x, z\} = U$, whence there are unique numbers α, β in \mathbb{R} such that $y = \alpha x + \beta z$. By (m2), we have $x + \alpha x \perp_\rho \beta z$ and therefore, from (4.3),

$$f(x + \alpha x + \beta z) - f(x + \alpha x) - f(\beta z) \in D.$$

Moreover, from the $((-D) + D + D)$ -additivity of f on one-dimensional subspaces, we get

$$f(x + \alpha x) - f(x) - f(\alpha x) \in (-D) + D + D$$

and since $\alpha x \perp_\rho \beta z$, it follows from (4.3) that

$$f(\alpha x + \beta z) - f(\alpha x) - f(\beta z) \in D.$$

Combining the last three statements we obtain (4.4) and the proof is complete. □

Notice that taking in the above theorem $D = \{0\}$, we immediately get the following generalization of Theorem 1.2 (however, we assume here commutativity of the target group).

Theorem 4.3. *Let X be a real normed linear space with $\dim X \geq 2$ and let G be an abelian group. A function $f: X \rightarrow G$ satisfies, with $\varepsilon \in (0, 1)$, the condition*

$$x \perp_\rho^\varepsilon y \implies f(x + y) = f(x) + f(y), \quad x, y \in X,$$

if and only if the function f is additive.

Consider now a real normed linear space Y as a target space and get another consequence of Theorem 4.2.

Theorem 4.4. *Let X and Y be real normed linear spaces and $\dim X \geq 2$. Suppose that, with the constants $\varepsilon \in (0, 1)$ and $\delta \geq 0$, a function $f: X \rightarrow Y$ satisfies the condition*

$$x \perp_\rho^\varepsilon y \implies \|f(x + y) - f(x) - f(y)\| \leq \delta, \quad x, y \in X.$$

Then

$$\|f(x + y) - f(x) - f(y)\| \leq 5\delta, \quad x, y \in X,$$

i.e., f is a 5δ -additive mapping.

Moreover, if Y is a Banach space, then there exists exactly one additive mapping $g: X \rightarrow Y$ such that $\|f(x) - g(x)\| \leq 5\delta$ for all $x \in X$.

Proof. Let $D := \{z \in Y : \|z\| \leq \delta\}$ be a closed ball in Y . Then $\tilde{D} = 5D = \{z \in Y : \|z\| \leq 5\delta\}$ and the first assertion follows from Theorem 4.2. Furthermore, if the space Y is complete, on account of the celebrated Hyers-Ulam Theorem (cf., [13, 14]) there exists a unique additive mapping g which suitably approximates f . \square

A comparison of the above Theorem 4.4 with Theorems 3.2 and 3.4 and their proofs, enable us to present two final results of the paper.

Theorem 4.5. *Let $(X_1, \|\cdot\|_1)$, $(X_2, \|\cdot\|_2)$ and $(Y, \|\cdot\|)$ be nontrivial real normed linear spaces. If a function $f: X_1 \oplus_\infty X_2 \rightarrow Y$ satisfies, with some $\delta \geq 0$,*

$$x \perp_\rho y \implies \|f(x + y) - f(x) - f(y)\| \leq \delta, \quad x, y \in X_1 \times X_2, \quad (4.5)$$

then

$$\|f(x + y) - f(x) - f(y)\| \leq 9\delta, \quad x, y \in X_1 \times X_2, \quad (4.6)$$

i.e., f is a 9δ -additive mapping. Moreover, if Y is a Banach space, then there exists an additive mapping $g: X_1 \oplus_\infty X_2 \rightarrow Y$ such that $\|f(x) - g(x)\| \leq 9\delta$ for all $x \in X_1 \times X_2$.

Proof. Analogously as in the proof of Theorem 3.2, we first approximate the Cauchy difference on the spaces $\{0\} \times X_2$ and $X_1 \times \{0\}$. Namely, for fixed $(0, x'_2), (0, x''_2) \in \{0\} \times X_2$ and $x_1 \in X_1$ such that $\|x_1\|_1 > \|x'_2\|_2$ we have by Lemma 3.1, $(x_1, 0) \perp_\rho (0, x'_2)$, $(x_1, x'_2) \perp_\rho (0, x''_2)$, $(x_1, 0) \perp_\rho (0, x'_2 + x''_2)$, whence,

$$\begin{aligned} \|f(x_1, x'_2) - f(x_1, 0) - f(0, x'_2)\| &\leq \delta, \\ \|f(x_1, x'_2 + x''_2) - f(x_1, x'_2) - f(0, x''_2)\| &\leq \delta, \\ \|f(x_1, x'_2 + x''_2) - f(x_1, 0) - f(0, x'_2 + x''_2)\| &\leq \delta, \end{aligned}$$

and therefore,

$$\|f(0, x'_2 + x''_2) - f(0, x'_2) - f(0, x''_2)\| \leq 3\delta. \quad (4.7)$$

In a similar way we get, for $x'_1, x''_1 \in X_1$,

$$\|f(x'_1 + x''_1, 0) - f(x'_1, 0) - f(x''_1, 0)\| \leq 3\delta. \quad (4.8)$$

Now, in order to show (4.6), fix $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$. Since $X_1 \times \{0\} \perp_\rho \{0\} \times X_2$ and $\{0\} \times X_2 \perp_\rho X_1 \times \{0\}$, we have by (4.5),

$$\begin{aligned} \|f(x_1 + y_1, x_2 + y_2) - f(x_1 + y_1, 0) - f(0, x_2 + y_2)\| &\leq \delta, \\ \|f(x_1, x_2) - f(x_1, 0) - f(0, x_2)\| &\leq \delta, \\ \|f(y_1, y_2) - f(y_1, 0) - f(0, y_2)\| &\leq \delta, \end{aligned}$$

which together with (4.7) and (4.8) gives

$$\|f(x_1 + y_1, x_2 + y_2) - f(x_1, x_2) - f(y_1, y_2)\| \leq 9\delta$$

and finishes the first part of the proof. The second part is obtained by applying the Hyers-Ulam theorem (as in the proof of Theorem 4.4). \square

Theorem 4.6. *Let Y be a real normed linear space and let $f: \mathcal{C}[0, 1] \rightarrow Y$ satisfy with some $\delta \geq 0$ the conditional inequality*

$$\varphi \perp_{\rho} \psi \implies \|f(\varphi + \psi) - f(\varphi) - f(\psi)\| \leq \delta, \quad \varphi, \psi \in \mathcal{C}[0, 1]. \quad (4.9)$$

Then

$$\|f(\varphi + \psi) - f(\varphi) - f(\psi)\| \leq 21\delta, \quad \varphi, \psi \in \mathcal{C}[0, 1]. \quad (4.10)$$

Moreover, if Y is a Banach space, then there exists an additive mapping $g: \mathcal{C}[0, 1] \rightarrow Y$ such that $\|f(\varphi) - g(\varphi)\| \leq 21\delta$ for all $\varphi \in \mathcal{C}[0, 1]$.

Proof. We only sketch the proof, keeping the notations from the proof of Theorem 3.4. We have $\varphi_1 \notin \ker \eta_{t_1}$, $\text{span}\{\varphi_1\} \subseteq \ker \eta_{t_2}$, $\varphi_1 \perp_{\rho} \ker \eta_{t_1}$ and

$$\mathcal{C}[0, 1] = \text{span}\{\varphi_1\} + \ker \eta_{t_1}.$$

Fix $\varphi, \psi \in \mathcal{C}[0, 1]$. There exist $\alpha, \beta \in \mathbb{R}$ and $h, p \in \ker \eta_{t_1}$ such that $\varphi = \alpha\varphi_1 + h$ and $\psi = \beta\varphi_1 + p$. On account of (4.9), we have

$$\begin{aligned} \|f(\varphi + \psi) - f(\alpha\varphi_1 + \beta\varphi_1) - f(h + p)\| &\leq \delta, \\ \|f(\varphi) - f(\alpha\varphi_1) - f(h)\| &\leq \delta, \\ \|f(\psi) - f(\beta\varphi_1) - f(p)\| &\leq \delta. \end{aligned}$$

Since $\ker \eta_{t_1}$ is isometrically isomorphic to $M_{t_1} \oplus_{\infty} N_{t_1}$, in view of Theorem 4.5, mappings $f|_{\ker \eta_{t_1}}$ and $f|_{\ker \eta_{t_2}}$ satisfy (4.6), whence

$$\begin{aligned} \|f(h + p) - f(h) - f(p)\| &\leq 9\delta, \\ \|f(\alpha\varphi_1 + \beta\varphi_1) - f(\alpha\varphi_1) - f(\beta\varphi_1)\| &\leq 9\delta \end{aligned}$$

and all the above five inequalities yield (4.10). The second part of the theorem is obtained in a usual manner. \square

5. Concluding Remarks

An analysis of the results obtained in the paper leads to a distinction of two cases: the exact ρ -orthogonality \perp_{ρ} and the essentially approximate ρ -orthogonality $\perp_{\rho}^{\varepsilon}$ (with strictly positive ε). Comparing Theorems 1.1 and 1.2, we see that for a conditional functional equation, just a mild enlarging of the set of admissible elements may result in a substantial change of the solution. This fact allows us to look at the results obtained in the paper from a better perspective. In Sect. 3, working with $\varepsilon = 0$, we are able to get solutions of (3.1) merely in some specific (though vast) classes of normed linear spaces while the question about the general solution of (3.1) remains open (Problem 3.7). We emphasize that in Sect. 3 we do not assume commutativity of G and we apply our results, in particular, to Birkhoff-James orthogonally additive mappings (Theorem 3.3). It remains an unanswered question whether Theorem 3.5 is valid also for non-abelian groups.

Contrary to Sect. 3, most of the results in Sect. 4 are obtained for $\varepsilon > 0$, although with no need of additional assumptions on the domain, which could

be an arbitrary normed linear space. We should admit, however, that also the target space is a normed linear one, whence the commutativity of values is guaranteed. The natural question in this context is whether Theorem 4.3 works also for a non-abelian group G .

At the end of Sect. 4, considering the stability of (3.1), we came back to the case $\varepsilon = 0$. In Theorems 4.5 and 4.6, setting $\delta = 0$, we obtain orthogonally additive mappings in the relevant spaces. However, the results of Sect. 3 are still stronger, since in corresponding Theorems 3.2 and 3.4 the commutativity of the group G is not assumed.

Finally, let us notice that all the results of the paper which concern the ρ -orthogonality can be also considered for ρ_{+-} and ρ_{-} -orthogonality relations

$$x \perp_{\rho_+} y \iff \rho'_+(x, y) = 0 \quad \text{and} \quad x \perp_{\rho_-} y \iff \rho'_-(x, y) = 0.$$

Notice that, unlike ρ' , the functions ρ'_+ and ρ'_- need not be homogeneous (indeed, non-smooth normed linear spaces furnish the necessary examples) and therefore the relations $\perp_{\rho_+}^\varepsilon$ and $\perp_{\rho_-}^\varepsilon$ need not satisfy (ort-a). Also the notion of an approximate ρ -orthogonality can be easily adapted to *approximate* ρ_{+-} and ρ_{-} -orthogonalities given by $x \perp_{\rho_\pm}^\varepsilon y \iff |\rho'_\pm(x, y)| \leq \varepsilon \|x\| \|y\|$ (see [7, 8]).

Studying then the classes of exact and approximate ρ_{+-} and ρ_{-} -orthogonally additive mappings, analogous results to those presented in this paper can be obtained (despite of the lack of homogeneity of ρ'_+ and ρ'_-). And also in that case the results are essentially new, i.e., not derivable from earlier works.

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Jacek Chmieliński and Paweł Wójcik
Department of Mathematics
Pedagogical University of Krakow
Podchorążych 2
Kraków, 30-084
Poland
e-mail: jacek.chmielinski@up.krakow.pl;
pawel.wojcik@up.krakow.pl

Justyna Sikorska
Institute of Mathematics
University of Silesia
Bankowa 14
Katowice, 40-007
Poland
e-mail: justyna.sikorska@us.edu.pl

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