Title: Delta-convexity with given weights

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DELTA-CONVEXITY WITH GIVEN WEIGHTS

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Dedicated to Professor Zygfryd Kominek on the occasion of his 75th birthday

Abstract. Some differentiability results from the paper of D.Ş. Marinescu & M. Monea [7] on delta-convex mappings, obtained for real functions, are extended for mappings with values in a normed linear space. In this way, we are nearing the completion of studies established in papers [2], [5] and [7].

1. Motivation and main results

While solving Problem 11641 posed by a Romanian mathematician Nicolae Bourbăcuț in [2] I was announcing in [5] (without proof) the following

Theorem 1.1. Assume that we are given a differentiable function $\varphi$ mapping an open real interval $(a, b)$ into the real line $\mathbb{R}$. Then each convex solution $f: (a, b) \rightarrow \mathbb{R}$ of the functional inequality

\[
(*) \quad \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right) \leq \frac{\varphi(x) + \varphi(y)}{2} - \varphi\left(\frac{x + y}{2}\right), \quad x, y \in (a, b),
\]

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is differentiable and the inequality

\[ |f'(x) - f'(y)| \leq |\varphi'(x) - \varphi'(y)| \]

holds true for all \( x, y \in (a, b) \).

The proof reads as follows.

Put \( g := f - \varphi \). Then (\( ) \) states nothing else but the Jensen concavity of \( g \), i.e.

\[ \frac{1}{2}g(x) + \frac{1}{2}g(y) \leq g\left(\frac{x + y}{2}\right), \quad x, y \in (a, b). \]

It is widely known that a continuous Jensen concave function is concave in the usual sense. Since \( f \) itself is continuous (as a convex function on an open interval) and \( \varphi \) is differentiable then, obviously, our function \( g \) is continuous and hence concave. In particular, the one-sided derivatives of \( g \) do exist on \( (a, b) \) and we have

\[ g'_+(x) \leq g'_-(x) \quad \text{for all} \quad x \in (a, b). \]

Therefore

\[ f'_+(x) = g'_+(x) + \varphi'(x) \leq g'_-(x) + \varphi'(x) = f'_-(x) \leq f'_+(x) \]

for all \( x \in (a, b) \) because of the convexity of \( f \), which proves the differentiability of \( f \) on \( (a, b) \).

To show that \( f \) satisfies the assertion inequality, observe that whenever \( x, y \in (a, b) \) are such that \( x \leq y \), then

\[
|f'(x) - f'(y)| = f'(y) - f'(x) = g'(y) + \varphi'(y) - g'(x) - \varphi'(x)
\leq \varphi'(y) - \varphi'(x) = |\varphi'(x) - \varphi'(y)|,
\]

because the derivative of a differentiable convex (resp. concave) function is increasing (resp. decreasing). In the case where \( y \leq x \) it suffices to interchange the roles of the variables \( x \) and \( y \) in the latter inequality, which completes the proof.

Note that the convexity assumption imposed upon \( f \) in the above result renders (\( ) \) to be equivalent to

\[ \frac{|f(x) + f(y)|}{2} - f\left(\frac{x + y}{2}\right) \leq \frac{\varphi(x) + \varphi(y)}{2} - \varphi\left(\frac{x + y}{2}\right), \quad x, y \in (a, b), \]
defining (in the class of continuous functions) the notion of delta convexity in the sense of L. Veselý and L. Zajíček (see [10]).

In that connection, D.Ş. Marinescu and M. Monea have proved, among others, the following result (see [7, Theorem 2.7]).

**Theorem M-M.** Let \( \varphi: (a,b) \longrightarrow \mathbb{R} \) be a differentiable function and let \( f: (a,b) \longrightarrow \mathbb{R} \) be a convex function admitting some scalars \( s,t \in (0,1) \) such that the inequality

\[
 tf(x) + (1 - t)f(y) - f(sx + (1 - s)y) 
\]

\[
 \leq t\varphi(x) + (1 - t)\varphi(y) - \varphi(sx + (1 - s)y)
\]

is satisfied for all \( x,y \in (a,b) \). Then the function \( f \) is differentiable and the inequality

\[
|f'(x) - f'(y)| \leq |\varphi'(x) - \varphi'(y)|
\]

holds true for all \( x,y \in (a,b) \).

Without any convexity assumption we offer the following counterpart of Theorem M-M for vector valued mappings.

**Theorem 1.2.** Given an open interval \( (a,b) \subset \mathbb{R} \), a normed linear space \( (E, \| \cdot \|) \), and two real numbers \( s,t \in (0,1) \) (weights) assume that a map \( F: (a,b) \longrightarrow E \) is delta \((s,t)\)-convex with a differentiable control function \( f: (a,b) \longrightarrow \mathbb{R} \), i.e. that a functional inequality

\[
\|tF(x) + (1 - t)F(y) - F(sx + (1 - s)y)\|
\]

\[
\leq tf(x) + (1 - t)f(y) - f(sx + (1 - s)y)
\]

is satisfied for all \( x,y \in (a,b) \). If the function

\( (a,b) \ni x \mapsto \|F(x)\| \in \mathbb{R} \)

is upper bounded on a set of positive Lebesgue measure, then \( F \) is differentiable and the inequality

\[
\|F'(x) - F'(y)\| \leq |f'(x) - f'(y)|
\]

holds true for all \( x,y \in (a,b) \).
**Corollary.** Under the assumptions of Theorem 1.2, the vector valued map $F$ is continuously differentiable.

**Proof.** Fix arbitrarily an $x \in (a, b)$ and $h \in \mathbb{R}$ small enough to have $x + h \in (a, b)$ as well. Then

$$\|F'(x + h) - F'(x)\| \leq |f'(x + h) - f'(x)|$$

and the right-hand side difference tends to zero as $h \to 0$ because a differentiable convex function is of class $C^1$. □

The assumption that the function $(a, b) \ni x \mapsto \|F(x)\| \in \mathbb{R}$ is upper bounded on a set of positive Lebesgue measure, may be replaced by numerous alternative conditions forcing a scalar Jensen convex function on $(a, b)$ to be continuous.

**Theorem 1.3.** Given an open interval $(a, b) \subset \mathbb{R}$, a normed linear space $(E, \| \cdot \|)$ that is reflexive or constitutes a separable dual space, and two weights $s, t \in (0, 1)$, assume that a map $F: (a, b) \to E$ is delta $(s, t)$-convex with a $C^2$-control function $f: (a, b) \to \mathbb{R}$. If the function

$$(a, b) \ni x \mapsto \|F(x)\| \in \mathbb{R}$$

is upper bounded on a set of positive Lebesgue measure, then $F$ is twice differentiable almost everywhere in $(a, b)$ and the domination

$$\|F''(x)\| \leq f''(x)$$

holds true for almost all $x \in (a, b)$.

The assumption that a normed linear space $(E, \| \cdot \|)$ spoken of in Theorem 1.3 is reflexive or constitutes a separable dual space may be replaced by a more general requirement that $(E, \| \cdot \|)$ has the Radon-Nikodym property (RNP), i.e. that every Lipschitz function from $\mathbb{R}$ into $E$ is differentiable almost everywhere. This definition (of Rademacher type character) is not commonly used but is more relevant to the subject of the present paper. R.S. Phillips [9] showed that reflexive Banach spaces enjoy the RNP whereas N. Dunford and B.J. Pettis [3] proved that separable dual spaces have the RNP.
2. Proofs

To prove Theorem 1.2 we need the following

**Lemma.** Given weights \( s, t \in (0, 1) \) assume that a map \( F : (a, b) \rightarrow E \) is delta \((s, t)\)-convex with a control function \( f : (a, b) \rightarrow \mathbb{R} \). Then the inequality

\[
\| \lambda F(x) + (1 - \lambda) F(y) - F(\lambda x + (1 - \lambda) y) \| \leq \lambda f(x) + (1 - \lambda) f(y) - f(\lambda x + (1 - \lambda) y)
\]

is valid for all \( x, y \in (a, b) \) and every rational \( \lambda \in (0, 1) \). In particular, \( F \) is delta Jensen convex with a control function \( f \), i.e. the inequality

\[
\left\| \frac{F(x) + F(y)}{2} - F\left(\frac{x + y}{2}\right) \right\| \leq \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right)
\]

holds true for all \( x, y \in (a, b) \).

**Proof.** Fix arbitrarily a continuous linear functional \( x^* \) from the unit ball in the dual space \( E^* \). Then the delta \((s, t)\)-convexity of \( F \) implies that for all \( x, y \in (a, b) \) one has

\[
t(x^* \circ F)(x) + (1 - t)(x^* \circ F)(y) - (x^* \circ F)(sx + (1 - s)y)
\leq tf(x) + (1 - t)f(y) - f(sx + (1 - s)y)
\]

or, equivalently,

\[
(f - x^* \circ F)(sx + (1 - s)y) \leq t(f - x^* \circ F)(x) + (1 - t)(f - x^* \circ F)(y).
\]

By means of Theorem 3 from N. Kuhn’s paper [6] we deduce that the function \( g := f - x^* \circ F \) enjoys the convexity type property

\[
g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y), \quad x, y \in (a, b), \lambda \in (0, 1) \cap \mathbb{Q},
\]

where \( \mathbb{Q} \) stands for the field of all rationals. Consequently, for all \( x, y \in (a, b) \) and \( \lambda \in (0, 1) \cap \mathbb{Q} \), we get the inequality

\[
\lambda(x^* \circ F)(x) + (1 - \lambda)(x^* \circ F)(y) - (x^* \circ F)(\lambda x + (1 - \lambda)y)
\leq \lambda f(x) + (1 - \lambda)f(y) - f(\lambda x + (1 - \lambda)y).
\]
Replacing here the functional \( x^* \) by \(-x^*\) we infer that \textit{a fortiori}
\[
|x^*(\lambda F(x) + (1 - \lambda)F(y) - F(\lambda x + (1 - \lambda)y))| \\
\leq \lambda f(x) + (1 - \lambda)f(y) - f(\lambda x + (1 - \lambda)y),
\]
which due to the unrestricted choice of \( x^* \) gives the assertion desired. \( \square \)

\textbf{Remark 2.1.} Using another method, A. Olbryś ([8, Lemma 1]) with the aid of the celebrated Daróczy and Páles identity
\[
\frac{x + y}{2} = s\left[\frac{s x + y}{2} + (1 - s)y\right] + (1 - s)\left[ sx + (1 - s)\frac{x + y}{2}\right],
\]
has proved that any delta \((s,t)\)-convex map on a convex subset of a real Banach space is necessarily delta Jensen convex.

\textbf{Proof of Theorem 1.2.} In view of the Lemma, \( F \) is delta Jensen convex with a control function \( f \). Due to the differentiability of \( f \) and the regularity assumption upon \( F \) the map
\[
(a, b) \ni x \mapsto f(x) + \|F(x)\| \in \mathbb{R}
\]
is upper bounded on a set of positive Lebesgue measure. Thus, with the aid of author’s result from [4], we obtain the local Lipschitz property of \( F \) and, in particular, the fact that \( F \) is a delta convex map controlled by \( f \) in the sense of L. Veselý & L. Zajíček (see [10]). Therefore, for any member \( x^* \) from the unit ball in the dual space \( E^* \) the function \( g_* := f + x^* \circ F \) is convex. Moreover, on account of Proposition 3.9 (i) in [10] p. 22] (see also Remark 2.2 below), \( F \) yields a differentiable map. Hence, \( g_* \) is differentiable as well and the derivative \( g'_* \) is increasing. Consequently, for any two fixed elements \( x, y \in (a, b), x \leq y \), we obtain the inequality
\[
(x^* \circ F)'(x) - (x^* \circ F)'(y) = g'_*(x) - f'(x) - g'_*(y) + f'(y) \\
\leq -f'(x) + f'(y) \leq |f'(x) - f'(y)|.
\]
Replacing here the functional \( x^* \) by \(-x^*\) we arrive at
\[
|x^*(F'(x) - F'(y))| = |(x^* \circ F)'(x) - (x^* \circ F)'(y)| \leq |f'(x) - f'(y)|,
\]
which, due to the unrestricted choice of \( x^* \) from the unit ball in \( E^* \), implies that
\[
\|F'(x) - F'(y)\| \leq |f'(x) - f'(y)|.
\]
In the case where \( y \leq x \) it suffices to interchange the roles of \( x \) an \( y \) in the latter inequality. Thus the proof has been completed.  

\[ \square \]

**Remark 2.2.** Actually, Proposition 3.9 (i) in \( [10] \) p. 22] states that \( F \) is even strongly differentiable at each point \( x \in (a, b) \), i.e. for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) and an element \( c(x) \in E \) such that for all points \( u, v \in (x - \delta, x + \delta) \subset (a, b), u \neq v \), one has

\[
\left\| \frac{F(v) - F(u)}{v - u} - c(x) \right\| \leq \varepsilon.
\]

Obviously, every strongly differentiable map is differentiable (in general, in the sense of Fréchet).

**Proof of Theorem 1.3.** In view of Theorem 1.2 \( F \) is differentiable and the inequality

\[
\| F'(x) - F'(y) \| \leq | f'(x) - f'(y) |
\]

holds true for all \( x, y \in (a, b) \). Let a closed interval \([\alpha, \beta]\) be contained in \((a, b)\). Since, a continuously differentiable function, \( f'_{[\alpha, \beta]} \) yields an absolutely continuous function, for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that, for every finite collection of pairwise disjoint subintervals \((a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\) of \([\alpha, \beta]\) with \( \sum_{i=1}^{k} (b_i - a_i) < \delta \), one has \( \sum_{i=1}^{k} |f'(b_i) - f'(a_i)| < \varepsilon \), whence

\[
\sum_{i=1}^{k} \| F'(b_i) - F'(a_i) \| \leq \sum_{i=1}^{k} |f'(b_i) - f'(a_i)| < \varepsilon.
\]

This proves that the map \( F'_{[\alpha, \beta]} \) is absolutely continuous as well. Since the space \((E, \| \cdot \|)\) enjoys the Radon-Nikodym property, in virtue of Theorem 5.21 from the monograph \([1]\) by Y. Benyamini and J. Lindenstrauss, the map \( F''_{[\alpha, \beta]} \) is differentiable almost everywhere in \([\alpha, \beta]\), i.e. off some nullset \( T \subset [\alpha, \beta] \) the second derivative \( F''(x) \) of \( F \) at \( x \) does exist for all \( x \in [\alpha, \beta] \setminus T \).

Now, fix arbitrarily a strictly decreasing sequence \((\alpha_n)_{n \in \mathbb{N}}\) and a strictly increasing sequence \((\beta_n)_{n \in \mathbb{N}}\) such that \( a < \alpha_n < \beta_n < b, n \in \mathbb{N} \), convergent to \( a \) and \( b \), respectively. Then, for every \( n \in \mathbb{N} \) one may find a nullset \( T_n \subset [\alpha_n, \beta_n] \) such that the second derivative \( F''(x) \) of \( F \) at \( x \) does exist for all \( x \in [\alpha, \beta] \setminus T_n \). Putting \( T := \bigcup_{n \in \mathbb{N}} T_n \) we obtain a set of Lebesgue measure zero, contained in \((a, b)\), such that the second derivative \( F''(x) \) does exist for all points \( x \in (a, b) \setminus T \). Fix arbitrarily a point \( x \in (a, b) \setminus T \). Then for any point \( y \in (a, b) \setminus \{x\} \) we have

\[
\left\| \frac{F'(y) - F'(x)}{y - x} \right\| \leq \left| \frac{f'(y) - f'(x)}{y - x} \right|
\]
and passing to the limit as \( y \to x \) we conclude that
\[
\|F''(x)\| \leq |f''(x)| = f''(x),
\]
because of the convexity of \( f \), which completes the proof. \( \square \)

**Remark 2.3.** Theorem 5.21 from [1] states, among others, that any absolutely continuous map from the unit interval \([0,1]\) into a normed linear space \( E \) with the Radon-Nikodym property is differentiable almost everywhere. It is an easy task to check (an affine change of variables) that any absolutely continuous map on a compact interval \([\alpha,\beta] \subset \mathbb{R}\) with values in \( E \) is almost everywhere differentiable as well.

**References**


