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ZYGFRYD KOMINEK, A MATHEMATICIAN, A TEACHER, A FRIEND

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Dedicated to Professor Zygryd Kominek on his 75th birthday

1. Introduction

For the first time I heard about a young mathematician of the name Zygfryd Kominek was in early 1970's when I was a student of mathematics at the Silesian University in Katowice. I had friends studying physics at the same university, and one day I was told by one of them, that they had the calculus course with a guy who went in a military uniform. That was Zygfryd, who then performed military service, or rather a complement to it allowing him to be promoted to officer's level. Not that he wanted it but that was in times of communist Poland where the military service was mandatory (students used to perform it during Summer holidays), and fresh PhD's were proposed to make an additional service - this was the case of Dr. Kominek. Later I met him personally, first indirectly (I used to have classes with Bożena Szymura

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who became Mrs. Kominek soon), and finally directly - we attended the same seminar on functional equations, founded and lead by Marek Kuczma.

2. Mathematician

The papers that were authored or co-authored by Professor Kominek range from functional equations and inequalities, real functions, measure theory, general topology, functional analysis, probability theory and stochastic processes, group theory and convex geometry.

Soon after our meeting at the Kuczma's seminar I was permitted to adress him as "Zyga", and we became close co-workers. Actually, I was one of the co-authors of only one paper, the other was Roman Ger (I mean the paper [19]). Let me quote a fragment of the review written by S. Łojasiewicz (cf. [MR1]) *The authors generalize [M. Kuczma's and J. Smítal's, cf.* [KS]*] result in the following way: Let X be a metric space and μ^* an outer measure satisfying some natural conditions (which are always true for $\mathbb{R}^n$ and the Lebesgue measure). Let $E, D \subset X$, $x_0 \in E$ and a map $f : E \times X \to X$ be such that $f(x_0 \times D)$ is dense and $f(x_0, y)$ is an outer density point of $f(E \times y)$ in a uniform way with respect to y. Then $\mu^*(B \cap f(E \times D)) = \mu^*(B)$ for every ball B of X. Several applications are given and a topological analogue is proved.* Let us recall that the original Smítal's Lemma reads as follows.

Lemma 2.1. *If $E \subset \mathbb{R}$ has a positive outer measure and $D \subset \mathbb{R}$ is a dense set in $\mathbb{R}$, then the inner Lebesgue measure of the set $\mathbb{R} \setminus (E + D)$ is equal to zero.*

The paper [19] was cited rather often. It contains (in the section "Backgrounds") several examples of applications of the [M. Kuczma's and] J. Smítal's Lemma to prove classical theorems in the theory of functional equations in several variables. For instance, the celebrated Steinhaus theorem, stating that the Minkowski sum of two measurable subsets of the real line contains a non-degenerate interval, can be proved (and actually is) with the help of the Smítal's Lemma. In a similar way the theorem of Fréchet and Sierpiński about the form of any measurable solution of the Cauchy equation, can be also derived from Smítal's Lemma. Also, the authors of [19] proved, using Smítal's Lemma, that every microperiodic and measurable function mapping reals to reals has to be constant almost everywhere, and each non-measurable subspace L of $\mathbb{R}$ over $\mathbb{Q}$ has to be saturated non-measurable (i.e. the inner Lebesgue measure of both L and $\mathbb{R} \setminus L$ are 0). Let me for instance quote the proof of Fréchet-Sierpiński theorem, which states that any measurable and additive function $f\colon \mathbb{R} \to \mathbb{R}$ is of the form $f(x) = f(1)x$, $x \in \mathbb{R}$. Indeed, define $g(x) := f(x) - f(1)x$ for every $x \in \mathbb{R}$, and observe that $g(r) = 0$, $r \in \mathbb{Q}$. Suppose that $g \neq 0$, and denote $A := \{x \in \mathbb{R} : g(x) > 0\}$. Then $-A = \{x \in \mathbb{R} : g(x) < 0\}$ and $\ell_1(A) = \ell_1(-A)$. Suppose that $\ell_1(A \cup (-A)) > 0$. Then $\ell_1(A) > 0$, and we also have $A = A + \mathbb{Q}$. From the Smítal's Lemma we infer that

$$\ell_1(\mathbb{R} \setminus A) = \ell_1(\mathbb{R} \setminus (A + \mathbb{Q})) = 0.$$

Hence the set $A \cap (-A) = \mathbb{R} \setminus [(\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus (-A))]$ is of full measure. In particular, there is a $y \in A \cap (-A)$. We have $g(y) > 0$ and, since $-y \in A$ we also have $g(-y) = -g(y) > 0$, which yields a contradiction. Therefore $\ell_1(A) = \ell_1(-A) = 0$, in other words $g(x) = 0$ a.e., say $g(x) = 0$ whenever $x \in \mathbb{R} \setminus W$, where W is a nullset. As for an arbitrary $z \in \mathbb{R}$ also $W - z$ is a nullset, we can find an $h \in \mathbb{R} \setminus (W \cup (W - z))$. Thus, taking into account additivity of g we get

$$g(z) = g(z + h) - g(h) = 0 - 0 = 0,$$

which ends the proof in view of arbitrary choice of $z \in \mathbb{R}$.

These remarks make the sense of generalizing the classic Smítal's Lemma justified.

2.1. Single variable

Functional equations and inequalities are the main subject treated by Prof. Kominek. Among the papers cited by MathSciNet, we find about 40 devoted to the topic. Initially, he dealt with functional equations in a single variable, proving theorems on the existence and uniqueness of solutions of equations of the type

$$\varphi(x) = h(x, \varphi[f(x)]), \tag{2.1}$$

or its generalization to a system

$$\varphi_i(x) = h_i\left(x, \varphi_1(f_{i,1}(x)), \ldots, \varphi_n(f_{i,n}(x))\right), \quad i \in \{1, \ldots, n\}, \tag{2.2}$$

where $h_i, f_{i,j},\ i, j \in \{1, \ldots, n\}$, are given and φ_i, $i \in \{1, \ldots, n\}$ are unknown. In a number of papers, either written separately or jointly (with J. Matkowski) (cf. [4]–[6] and [8]–[13]) he studies the existence and uniqueness of convex solutions of (2.1) or the existence and uniqueness of solutions of class C^r

of (2.2), under some additional assumptions. In some cases the author proves also the continuous dependence of C^r solutions of systems of the type (2.2). In papers [12] and [13] the author is interested in applying (his own) extensions of well known fixed point theorems to show the existence of solutions of a system of functional equations or to prove solvability criterion of systems of the type

$$T_i(x_1, \ldots, x_n) = S_i(x_1, \ldots, x_n), \quad i \in \{1, \ldots, n\},$$

involving metric contractivity hypotheses of the type

$$\rho_i\left(S_i(x_1, \ldots, x_n), S_i(y_1, \ldots, y_n)\right) \leq \sum_{k=1}^{n} a_{ik}\rho_k\left(T_k(x_1, \ldots, x_n), T_k(y_1, \ldots, y_n)\right)$$

for all $(x_1, \ldots, x_n)$ and $(x_1, \ldots, x_n)$ from $A = A_1 \times \cdots \times A_n$, and all $i \in \{1, \ldots, n\}$. Here we assume that absolute values of the eigenvalues of the matrix $[a_{ik}]$ are less than 1.

2.2. Steinhaus and Piccard type results

Investigating functional equations in a single variable did not prevent Zygfryd Kominek from getting involved in the research concerning functional equations in several variables and related issues. Actually, the very first paper by him, which was published in prestigious journal *Fundamenta Mathematicae* (cf. [1]) concerns a topological counterpart of the celebrated Steinhaus theorem. It is not necessary to explain what is the importance of this theorem for development of the theory of functional equations in several variables. This way Kominek started his adventure with Steinhaus-type theorems, one of his specialities during the career. Here is a result from [1]:

Theorem 2.1. *Let X be a topological vector space over a topological field K. If A, $B \subset X$ are second category Baire sets then $A+B$ contains a nonempty open set.*

From the result the well known theorem of S. Piccard (cf. [SP]) follows, and its version for Banach spaces was earlier proved by W. Orlicz and Z. Ciesielski (cf. [OC]).

In [1] the author mentions classes of sets connected with Jensen inequality or Cauchy equation. Investigation of such set classes is one of the leading subjects of Zygfryd Kominek research activity. For instance, in the paper [3] Kominek proves the following generalization of a result of Marcin E. Kuczma and Marek Kuczma from [KK].

THEOREM 2.2. *Let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that with respect to each variable h is a homeomorphism. If A, B are second cathegory Baire subsets of the real line then the set $h(A \times B)$ contains a [non-degenerate] interval.*

In a number of papers, written by himself or co-authored, Zygfryd Kominek was busy with different aspects of sums or differences of sets. During the Ger–Kominek Workshop in Mathematical Analysis and Real Functions, held in Katowice in 2015, Władysław Wilczyński presented a talk dedicated to Kominek's achievements in this area (cf. [WW]). He mentioned in particular the following papers: [1], [3], [15], [16], [18], [19], [23] and [49]. Let me add to this list pretty general papers [28] and [68]. In [15] the author proves the following theorem.

THEOREM 2.3. *There exist disjoint sets $A, B \subset \mathbb{R}$ such that $A \cup B = \mathbb{R}$ and such that*
(1) *A is a second Baire category set,*
(2) *B has infinite Lebesgue measure,*
(3) *$A + B$ does not contain any non-degenerate interval.*

PROOF. Take C – an arbitrary compact and nowhere dense set with positive Lebesgue measure. Put $B := (\mathbb{Q} - C) \cup (\mathbb{Q} + C)$. Then $B \supset \mathbb{Q} + C$, and in view of the Smítal's Lemma B has infinite Lebesgue measure ((2)). Moreover, we have $\mathbb{Q} - B \subset B$ whence, if we put $A := \mathbb{R} \setminus B$, follows (3). We also have (1), since B is a countable union of nowhere dense sets. $\square$

A generalization in a sense of the above theorem is the result proved in [16].

THEOREM 2.4. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be such that $f(x, \cdot)$ and $f(\cdot, y)$ are homeomorphisms for each real x, y. Then for any first category set A_0 with positive Lebesgue measure there exist subsets A and B of $\mathbb{R}$ such that*
(i) $A \cap B = \emptyset$, $A \cup B = \mathbb{R}$,
(ii) $A_0 \subset B$,
(iii) *B is of the first category with positive Lebesgue measure,*
(iv) $\mathrm{Int} f(A \times B) = \emptyset$.

In the same paper we find a generalization of a theorem by Harry I. Miller from [HM]. In early eighties of the last century both gentlemen met and started a fruitful cooperation. Harry visited Katowice and Zyga went to Sarajevo. They published two joint papers: [18] and [32]. In [18] the following theorem was proved.

THEOREM 2.5. *If A, $B \subset \mathbb{R}^n$, with A having a positive Lebesgue n-dimensional measure, and B having a positive n-dimensional outer Lebesgue measure then $A + B$ contains an n-dimensional ball.*

The proof is elementary and straightforward, but too long to be reproduced here. Let us note that the authors made a remark that their result is the best possible. Namely, if we assume that both A and B are non-measurable and have positive outer measure then $\mathrm{Int}(A + B)$ might be empty. Indeed, let us take an arbitrary Hamel basis H and let $h_0 \in H$ be fixed. Let T be a subspace spanned by $H \setminus \{h_0\}$ over $\mathbb{Q}$. If we put $C := T \cap (0, 1)$ then C is non-measurable with $\ell_1^*(C) = 1$, which follows e.g. from [AA]. Setting $A = B = C$, we get that $\mathrm{Int}(A + B) = \emptyset$.

This is a measure theoretical version of the following topological theorem by Wolfgang Sander (cf. [WS1]).

THEOREM 2.6. *If A, $B \subset \mathbb{R}^n$, and A is of second category Baire set and B is of second category, then $A + B$ contains an n-dimensional ball.*

Yet another result by W. Sander (cf. [WS2]) is reproduced in [18] but an elementary and straightforward proof is given.

THEOREM 2.7. *Suppose $A, B \subset \mathbb{R}$ and that A is measurable with $\ell_1(A) > 0$, $\ell_1^*(B) > 0$. Suppose further that H is an open set containing $A \times B$ and that f is a real valued function defined on H. If $a \in A$ is a density point of A, $b \in B$ is an outer density point of B and the partial derivatives f_x, f_y of f are continuous in some neighbourhood of (a, b) with $f_x(a, b) \neq 0 \neq f_y(a, b)$, then $f(A \times B)$ contains an interval.*

In the paper [23] we find the following result, which we quote after a review due to H.I. Miller (cf. [MR2]): *Assume that* (1) $(X; +)$ *is a separable Baire topological group (not necessarily commutative);* (2) μ *is a measure defined on some σ-algebra $\mathcal{M}$ of subsets of X such that each subset of positive measure contains a compact subset of positive measure. The main result of the paper is the following theorem: Let assumptions* (1) *and* (2) *be fulfilled. Then the following conditions are equivalent:* (3) *If $E \in \mathcal{M}$, $\mu(E) > 0$ and D is a countable dense subset of X, then $\mu(X \setminus (E + D)) = 0$.* (4) *If $A, B \in \mathcal{M}$, $\mu(A) > 0$ and $\mu(B) > 0$, then the set $(-A + B)$ has nonempty interior. We remark that this very interesting theorem extends previously observed connections (see, e.g.,* [19]*) between the classical Steinhaus theorem and Smítal's lemma in $\mathbb{R}^n$.*

Zygfryd Kominek has been collaborating with many mathematicians. Some of them were already mentioned, and in connection with operations on sets he co-authored the paper [49] written jointly with W. Wilczyński. They proved, among others, the following.

Theorem 2.8. *Let* $A \subset \mathbb{R}^n$ *be a Lebesgue measurable set. If*

$$\limsup_{r\to\infty} \frac{\ell_n(A \cap K_p(0,r))}{\ell_n(K_p(0,r))} =: \lambda > \frac{1}{2}, \tag{2.3}$$

then $A - A = \mathbb{R}^n$.

It is worth to notice that the authors give an example showing that the assumption (2.3) is the best possible. Actually, they show that the set $T \subset \mathbb{R}^p$ given by

$$T = \bigcup_{k\in\mathbb{Z}} [2k, 2k+1) \times \mathbb{R}^{n-1}$$

satisfies (2.3) with $\lambda = \frac{1}{2}$ but $(T - T) \cap (2\mathbb{Z}+1) \times \mathbb{R}^{n-1} = \emptyset$, whence it easily follows that $T - T \neq \mathbb{R}^n$.

Let us mention here a result from [28], which concerns a purely algebraic situation, where instead of $\mathbb{R}$ or $\mathbb{R}^n$, one has a commutative group. Kominek states in [28] the following result.

Theorem 2.9. *Let* $(G, +)$ *be a commutative group, divisible by* 2. *Then for every* $A \subset G$ *either* $A + A = G$ *or* $(G \setminus A) + A = G$ *or* $(G \setminus A) + (G \setminus A) = G$.

As a corollary he gets the following.

Theorem 2.10. *Let* A *be an arbitrary subset of a commutative and divisible by* 2 *group* G. *Then* $(G \setminus A) + A = G$ *if and only if for every* $y \in G$ *we have* $y - A \neq A - y$.

In the sequel of [28] Zygfryd is dealing with differences of subsets of commutative groups. It turns out that the situation is slightly different. In particular the following holds

Theorem 2.11. *Let* A *be an arbitrary subset of a commutative group* G *with the properties* $A - A \neq G$ *and* $(G \setminus A) - (G \setminus A) \neq G$. *Then* $A - A = (G \setminus A) - (G \setminus A)$ *and* $(G \setminus A) = A + x$ *iff* $x \notin A - A$. *If, moreover,* G *is a topological group then in particular the condition* $\mathrm{Int}(A - A) = \mathrm{Int}((G \setminus A) - (G \setminus A)) = \emptyset$ *implies* $\mathrm{Int}(A - (G \setminus A)) = \emptyset$.

The paper deals also with a mixed case. We have

Theorem 2.12. *Let* G *be an arbitrary commutative group divisible by* 2 *and let* $A \subset G$. *If* $A - A \neq G$ *and* $(G \setminus A) - (G \setminus A) \neq G$, *then* $A + A = (G \setminus A) + (G \setminus A) = (G \setminus A) + A = G$.

The paper [28] is concluded with yet another result.

Theorem 2.13. *Let G be an arbitrary commutative group divisible by* 2 *and let $A \subset G$. If $A - A \neq G$ then $(G \setminus A) + (G \setminus A) = G$.*

I am devoting pretty much place to the paper [28]. The reason for it is that in [28] the author shows his mastery, or even virtuosity of inventing the counterexamples, mostly based on specific constructions of Hamel bases. Zygfryd has been known for this skill, and I remember that during the seminars or international conferences he was answering questions just by producing a suitable example.

2.3. Classes of sets

Let us take a look on papers connected to some important classes of sets. We mean the papers [7] (written jointly with Bożena Kominek), [14] and [25], the latter written jointly with Roman Ger, Zygfryd's peer: they studied at the Jagiello University, Branch in Katowice. In [25] the following definition was recalled. If X is a linear topological Baire space then

$$\begin{aligned}\mathcal{B}(X) = \{T \subset X : &\text{ every additive functional defined on } X \\ &\text{ and bounded from above on } T \text{ is continuous}\}.\end{aligned} \tag{2.4}$$

In the same paper we find the following definition.

$$\begin{aligned}\mathcal{A}(X) = \{T \subset X : &\text{ for any open and convex set } D \supset T \\ &\text{ every Jensen-convex functional defined on } D \\ &\text{ and bounded from above on } T\text{,is continuous}\}.\end{aligned} \tag{2.5}$$

The definition of both classes was introduced by R. Ger and M. Kuczma in [GK]. In [MEK] Marcin E. Kuczma proved that

$$\mathcal{A}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}^n) \tag{2.6}$$

for each $n \in \mathbb{N}$. In the paper [25] the authors prove the following theorem, in a completely different way than that given by M.E. Kuczma in [MEK]. Actually, they had to elaborate a new method of proving the result, because M.E. Kuczma's method relied on the finite dimension of the domain (see the comment below).

THEOREM 2.14. *The equality $\mathcal{A}(X) = \mathcal{B}(X)$ holds for any real linear topological Baire space.*

While M.E. Kuczma to prove (2.6) used mainly a generalization of the fundamental Hahn-Banach extension theorem, Ger and Kominek apply rather results by Gerd Rodé from [GR] and a refinement of a theorem by Norbert Kuhn (cf. [NK]).

In the papers [7] and [14] the authors introduce the following classes of subsets of $\mathbb{R}^n$, with $n = 1$ in the case of [7],

$$\begin{aligned}\mathcal{B}_C^n = \{T \subset \mathbb{R}^n : &\text{ every additive function } f\colon \mathbb{R}^n \to \mathbb{R}\\ &\text{such that } f|T \text{ is continuous, is continuous}\},\\ \mathcal{A}_C^n = \{T \subset \mathbb{R}^n : &\text{ for any open and convex set } D \supset T\\ &\text{every Jensen convex function } f\colon D \to \mathbb{R}\\ &\text{such that } f|T \text{ is continuous, is continuous}\}.\end{aligned}$$

In both papers relations between classes $\mathcal{A}(\mathbb{R}^n)$, $\mathcal{B}(\mathbb{R}^n)$, $\mathcal{A}_C^n$, and $\mathcal{B}_C^n$ are studied. In [7] the authors give examples which show that for $n = 1$

- $\mathcal{B}_C^1 \setminus \mathcal{A}_C^1 \neq \emptyset$,
- $\mathcal{B}_C^1 \setminus \mathcal{B}(\mathbb{R}) \neq \emptyset$,
- $\mathcal{B}(\mathbb{R}) \setminus \mathcal{B}_C^1 \neq \emptyset$,
- $\mathcal{A}(\mathbb{R}) \setminus \mathcal{A}_C^1 \neq \emptyset$.

They prove however that

THEOREM 2.15. $\mathcal{A}_C^1 \subset \mathcal{A}$.

Thus we see that a slight change of the definition implies rather big differences between classes. In the paper [14] Kominek describes properties of classes $\mathcal{A}_C^n$ and $\mathcal{B}_C^n$. In particular he proves the following.

THEOREM 2.16. *If $g\colon [a,b] \to \mathbb{R}$ is continuous and not affine then the set*

$$I_g := \{(x,y) \in \mathbb{R}^2 : x \in [a,b] \wedge y = g(x)\}$$

belongs to the class $\mathcal{A}_C^2$.

In other words continuity of the restriction of a Jensen convex function $f : D \to \mathbb{R}$ where $\mathbb{R}^2 \supset D$ to the graph of a non-affine continuous function $g : [a,b] \to \mathbb{R}$ implies it continuity. In the proof uses the fact that I_g is compact and f has to be upper bounded on the set $\frac{1}{2}(I_g + I_g)$ which has a positive measure (see [MK]). It follows that f is continuous.

2.4. Semilinear topology

In 1980's Zygfryd was visiting rather often Marek Kuczma who was then very ill from the physical point of view but extremely healthy mentally. Kuczma was very active mathematically, and Kominek was co-author of four papers at the time. These were papers [26], [30], [36] and [39]. It is natural that we join here the paper [31] which was actually the habilitation dissertation of Zygfryd. The basic idea which appears in [26] is the concept of the *semilinear topology.* Let X be a linear space over a field $\mathbb{K}$ ($\mathbb{Q} \subset \mathbb{K} \subset \mathbb{R}$) endowed with a topology $\mathcal{T}$. Consider the function $\psi\colon \mathbb{K} \times X \times X \to X$ defined by

$$\psi(\lambda, x, y) = \lambda x + y$$

for every $\lambda \in \mathbb{K}$ and $x, y \in X$. If the function ψ is continuous with respect to $\mathcal{T}$ then we say that the topology $\mathcal{T}$ is linear, if it is separately continuous, we say that $\mathcal{T}$ is semilinear. The authors of [26] prove that $\mathcal{T}$ is semilinear if, and only if, for every fixed $\lambda \neq 0$ and $y \in X$ the function $\gamma\colon X \to X$ given by

$$\gamma(x) = \lambda x + y,$$

is continuous, and for every fixed $x, y \in X$ the function $\omega\colon \mathbb{K} \to X$ given by

$$\omega(\lambda) = \lambda x + y,$$

is continuous at 0. An example of semilinear topology is the so called *core topology.* The core topology is the topology of algebraically open subsets of X, i.e. such sets $A \subset X$ that $A = \mathrm{core}A$. By $\mathrm{core}A$ we understand the set of all points $y \in A$ which are *algebraically interior* to A. A point $y \in A$ is said to be algebraically interior to A if for every $x \in X$ there exists an $\varepsilon > 0$ such that

$$\lambda x + y \in A,$$

for every $\lambda \in (-\varepsilon, \varepsilon) \cap \mathbb{K}$. Recently, the $\mathrm{core}A$ is called *algebraic interior of A with respect to* $\mathbb{K}$ and denoted by $\mathrm{algint}_{\mathbb{K}}(A)$. If $\mathbb{K} = \mathbb{R}$ we simply write $\mathrm{algint}(A)$.

Denote by $\mathcal{T}(X)$ the core topology in the space X over $\mathbb{K}$. It turns out that $\mathcal{T}(X)$ is semilinear, and that for any semilinear topology $\mathcal{T}$ we have $\mathcal{T} \subset \mathcal{T}(X)$. The inclusion is strict if $\dim X > 1$. The concept of the core topology was very inspiring for many colleagues working in the functional equations theory. Zygfryd himself is the author of a brilliant result stating that *a midconvex functional is convex if and only if it is continuous in the core topology.* Actually this is a formulation remembered by his colleagues, an abbreviation expressing the great idea. This follows for instance from the following results contained in [31] (Theorem 3. 1 and Lemma 3.7, cf. also [26] and [30]). Let us recall that Kominek deals with the case where $\mathbb{K} = \mathbb{R}$.

THEOREM 2.17. *Let X be a linear space (over $\mathbb{R}$) endowed with a semilinear Baire topology (hence in particular with the core topology), let $D \subset X$ be an open and convex set, and let $f\colon D \to \mathbb{R}$ be a Jensen convex function. If f is lower semicontinuous in D then it is continuous in D.*

Without the assumption on Baire property, Zygfryd obtained the following (cf. Lemma 3.7 in [31]).

LEMMA 2.2. *Let X be a linear space endowed with a semilinear topology, let $D \subset X$ be an open and convex set, and let $f\colon D \to \mathbb{R}$ be a Jensen convex function. If f is lower semicontinuous in D then it is convex.*

The ideas connected with the core topology were inspiring for other authors. Let us mention the paper by K. Baron, M. Sablik and P. Volkmann [BSV], where continuity with respect to the core topology is a condition equivalent to the "decency"[1] of a character of a rational linear space. Actually we proved the following.

THEOREM 2.18. *A character of a rational linear space is decent if and only if it is continuous with respect to the core-topology.*

We say that a function $f\colon D \to \mathbb{R}$, where D is a convex subset of a linear space X, is Wright-convex if it satisfies the following inequality

$$f((1-\lambda)x + \lambda y) + f(\lambda x + (1-\lambda)y) \leq f(x) + f(y)$$

for every $x, y \in D$ and $\lambda \in [0,1]$. In [31] Kominek proved the following nice result, extending some earlier one obtained by C.T. Ng in [NG] where the final dimensional case is considered.

THEOREM 2.19. *Let X be a real linear space, and let $D \subset X$ be an algebraically open and convex set. A function $f\colon D \to \mathbb{R}$ is Wright-convex if and only if there exist a convex function $F\colon D \to \mathbb{R}$ and an additive function $a\colon X \to \mathbb{R}$ such that*

$$f(x) = F(x) + a(x),$$

for every $x \in D$.

[1] A character $\chi\colon G \to \mathbb{T}$ of a group G is "decent" if it has the representation $\chi(x) = \exp(2\pi i f(x))$, $x \in G$, with an additive $f\colon G \to \mathbb{R}$.

In [31] several results concerning classes $\mathcal{A}(X,\mathcal{T})$ and $\mathcal{B}(X,\mathcal{T})$ which are defined analogously to the classes given by (2.5) and (2.4), only here the notion of openness and continuity refer to the semilinear topology $\mathcal{T}$. In particular, the equality

$$\mathcal{A}(X,\mathcal{T}) = \mathcal{B}(X,\mathcal{T})$$

is proved in [31] under the assumption that $(X,\mathcal{T})$ is a Baire space. The result extends further the classical result by Marcin E. Kuczma (cf. [MEK] and [25], the papers cited above). Besides, in [31] we find extensions of several classical results, such as Bernstein-Doetsch and Mehdi theorems. Let us mention also paper [43], as example of description of additive functions continuous with respect to the algebraic topology.

Let us mention here a student of Zygfryd, Andrzej Olbryś who has been specializing in different variations of the classical convexity for many years. In particular, he got interested (or rather was advised to show interest) in the Wright-convexity by Zygfryd, while preparing his PhD dissertation. Andrzej Olbryś uses in his work the notion of algebraically open subsets of a linear space. Let us mention for instance the paper [AO] where the author deals with sets D which have nonempty core$D$ with respect to $\mathbb{Q}(t)$, t is a fixed number from $(0,1)$, (an extension of the field $\mathbb{Q}$ by the element t) or with respect to $\mathbb{R}$. Olbryś has extended the notion of convexity to the so called (ω,t)-convexity. If $D \subset X$ is t-convex, i.e. $(1-t)D + tD \subset D$ then a function $f\colon D \to \mathbb{R}$ is (ω,t)-convex iff

$$f((1-t)x + tz) \le (1-t)f(x) + tf(z) + \omega(x,z,t),$$

for every $x,z \in D$, where $\omega\colon D\times D\times[0,1] \to \mathbb{R}$ is a given function. A function $f\colon D \to \mathbb{R}$ is (ω,t)-concave iff

$$(1-t)f(x) + tf(z) + \omega(x,z,t) \le f((1-t)x + tz),$$

for every $x,z \in D$. If $f\colon D \to \mathbb{R}$ is (ω,t)-convex and (ω,t)-concave then we say that it is (ω,t)-affine. Let us fix a $t \in [0,1]$ and consider the following conditions that ω satisfies:

(a) $\omega(y,y,t) = 0$,
(b) $\omega(x,z,t) = \omega(z,x,(1-t))$,
(c) $(1-s)\omega(u,z,(1-s)) + s\omega(v,z,(1-s)) - \omega((1-s)u + sv, z,(1-s))$
$\le (1-s)\omega(u,v,(1-s)) - \omega((1-s)u + sz,(1-s)v + sz,(1-s))$,

for all $u,v,x,z \in D$, and $s \in \{t, 1-t\}$. Olbryś proved in particular the following theorem (cf. [AO, Theorem 7]).

Theorem 2.20. *Let D be a t-convex set and let* $\operatorname{algint}_{\mathbb{Q}(t)}(D) \neq \emptyset$. *Assume that $f\colon D \to \mathbb{R}$ is an (ω, t)-convex function where $\omega\colon D \times D \times [0,1] \to \mathbb{R}$. Then there exist a t-convex function $h\colon D \to \mathbb{R}$ and an (ω, t)-affine function $a\colon D \to \mathbb{R}$ such that*

$$f(x) = a(x) + h(x),\ x \in D,$$

if and only if ω satisfies the conditions (a)–(c) *for some point $y \in$* $\operatorname{algint}_{\mathbb{Q}(t)}(D)$.

2.5. Stability

Among the papers of Kominek, there is a number devoted to the questions of stability, i.e. problems asked originally by S. Ulam with respect to the additive functions, and first solved by D. Hyers, but then extended to a separate theory. Hyers and Ulam probably would not recognize the original question in the massive flood of results which multiply each year. And Zygfryd was here at the beginning of this brilliant career of the stability of functional equations and inequalities. Let us quote [29], [44], [45], [52], [58], [60], [61], [64], [65], [72], [76], and, to some extent, also [57].

Let us begin with recalling original question of S. Ulam (cf. [DH] and [HU]).

Problem 1. Let E and E' be Banach spaces and let δ be a positive number. A transformation $f(x)$ from E to E' will be called δ-linear if $\|f(x+y) - f(x) - f(y)\| < \delta$ for all x and y in E. Then the problem may be stated as follows. Does there exist for each $\epsilon > 0$ a $\delta > 0$ such that, to each δ-linear transformation $f(x)$ from E into E' there corresponds a linear transformation $\ell(x)$ from E into E' satisfying the inequality $\|f(x) - \ell(x)\| \leq \epsilon$ for all x in E?

The question was answered in affirmative by D.H. Hyers in [DH]. The question was then generalized in several directions, since this is not a survey of the vast area of stability we will not present all the issues. Let us concentrate on some selected results obtained by Zygfryd. In [65] the following theorem was proved.

Theorem 2.21. *Let S be an abelian semigroup, X be a Banach space, and let $g\colon S \to X$ be a function satisfying the following inequality*

$$\|g(x+2y) + g(x) - 2g(x+y) - 2g(y)\| \leq \omega(x,y), \quad x, y \in S^* = S \setminus \{0\},$$

where $\omega\colon S^* \times S^* \to \mathbb{R}$ fulfils conditions

$$\lim_{k\to\infty} 2^{-2k}\omega(2^k x, 2^k y) = 0, \quad x, y \in S^*,$$

$$\sum_{k=0}^{\infty} 2^{-2k}\omega(2^k x, 2^k y) \quad \textit{is convergent for all}$$

$$(u, v) \in \{(x, x), (x, 2x), (2x, x), (3x, x)\}, \quad x \in S^*.$$

Then there exists a unique function $f\colon S \to X$ *satisfying equation*

$$f(x + 2y) + f(x) = 2f(x + y) + 2f(y), \quad x, y \in S,$$

and the estimation

$$\|f(x) - g(x)\| \le \sum_{k=0}^{\infty} \frac{1}{4^{k+1}} \varphi(2^k x, 2^k x), \quad x \in S^*,$$

where $\varphi\colon S^* \times S^* \to \mathbb{R}$ *is given by*

$$\varphi(x, y) = \frac{1}{2}\left[\omega(x, y) + \omega(x, 2y) + \omega(3x, y) + 2\omega(2x, y)\right].$$

The most cited paper by Kominek (according to MathSciNet) is [29], where the author studies the local stability of Jensen equation. Let us quote again the review by László Losonczi (cf. [MR3]): *The author proves that Cauchy's functional equation and Jensen's functional equation are stable on some restricted domain* D *of* $\mathbb{R}^n_+$. *In the case of Cauchy's equation* D *is a bounded set such that* 0 *is in* intD, *the interior of* D, *and* $\frac{1}{2}D \subset D$. *In the case of Jensen's equation* D *is a bounded set with nonempty interior such that there exists an* $x_0 \in \mathrm{intD}$ *with* $\frac{1}{2}(D - x_0) \subset D - x_0$. *The result concerning Cauchy's equation extends that of F. Skof [Rend. Sem. Mat. Fis. Milano 53 (1983), 113–129; MR0858541].*

Let us note an interest in stability of the so called Jensen-Hosszú functional equation

$$f(x + y - xy) + f(xy) = 2f\left(\frac{x + y}{2}\right) \tag{2.7}$$

for all $x, y \in \mathbb{R}$. He proved in [67] the following.

Theorem 2.22. *Let* $\delta > 0$ *be a real constant and let* $g\colon \mathbb{R} \to \mathbb{R}$ *be a function satisfying the following inequality*

$$\left| g(x + y - xy) + g(xy) - 2g\left(\frac{x + y}{2}\right)\right| \le \delta \tag{2.8}$$

for all $x, y \in \mathbb{R}$. *Then there exists an additive function* $a\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$|g(x) - a(x) - g(0)| \le \frac{9}{2}\delta$$

for every $x \in \mathbb{R}$.

Note that the equation (2.7) arises when we take left-hand side of the Hosszú equation

$$f(x + y - xy) + f(xy) = f(x) + f(y) \tag{2.9}$$

and compare it with the right-hand side of the Jensen equation

$$f(x) + f(y) = 2f\left(\frac{x+y}{2}\right), \tag{2.10}$$

which hold in any set where the inside operations have sense. Both (2.9) and (2.10), in case of any real interval have the same solutions: $f(x) = a(x)+b$, where $a\colon \mathbb{R} \to \mathbb{R}$ is an additive function, and $b \in \mathbb{R}$ is a constant. Thus Theorem 2.22 expresses the fact that (2.7) is stable when treated for functions mapping reals to reals. Another natural domain for solutions of (2.9), (2.10) and (2.7) is the interval $I \in \{(0, 1), [0, 1]\}$. In this case however the behaviour of (2.10) and (2.9) is slightly different. Despite the fact that they remain equivalent, their stability properties differ. Actually, (2.10) remains stable in the sense of Hyers–Ulam, but (2.9) is not stable anymore (it was shown by Jacek Tabor in [JT]). Zygfryd, together with Justyna Sikorska wrote the paper [70], in which they proved the following.

Theorem 2.23. *Let* $\delta \ge 0$ *be given. If* $g\colon [0, 1] \to \mathbb{R}$ *satisfies* (2.8) *for* $x, y \in [0, 1]$ *then there exist an additive function* $a\colon \mathbb{R} \to \mathbb{R}$ *and a* $\mu \in \mathbb{R}$ *such that*

$$|g(x) - a(x) - g(0)| \le \mu\delta,$$

for all $x \in [0, 1]$.

Using a slightly different method (cf. [70]) they were able to prove also the following.

Theorem 2.24. *Let* $\delta \ge 0$ *be given. If* $g\colon (0, 1) \to \mathbb{R}$ *satisfies* (2.8) *for* $x, y \in (0, 1)$ *then there exist an additive function* $a\colon \mathbb{R} \to \mathbb{R}$, *and constants* $c \in \mathbb{R}$ *and* $\overline{\mu} \in \mathbb{R}$ *such that*

$$|g(x) - a(x) - c| \le \overline{\mu}\delta,$$

for all $x \in (0, 1)$.

Kominek continued the subject in [77], proving results for "pexiderized versions" of the Hosszú–Jensen equation. In particular, there is an example in [77] showing that in the most general case, i.e. inequality

$$\left|2f\left(\frac{x+y}{2}\right) - g(x+y-xy) - h(xy)\right| \leq \delta$$

which is fulfilled by functions f, g, and h mapping the unit interval I into $\mathbb{R}$, yet they are not "close" to affine mappings, as it could be expected. The positive results are obtained in cases where we restrict the number of functions to two (see Theorems 1–3 in [77]).

Let us also mention the paper [60], where the question of stability of the conditional Fréchet equation was studied (for increments from a subset C of a semigroup X).

Zygfryd wrote also a number of papers devoted to stability of inequalities. We have in mind the papers [21], [35], [61], [62]. For instance in [35] we find the following theorem.

THEOREM 2.25. *Let $(X, +)$ be a commutative, 2-divisible group. If $f\colon X \to \mathbb{R}$ is an ε-subadditive function (i.e. the inequality $f(x+y) \leq f(x) + f(y) + \varepsilon$ is satisfied for all $x, y \in X$) then there exists a Jensen function $h\colon X \to \mathbb{R}$ such that*

$$h(x) \leq f(x) \quad \textit{for every } x \in X.$$

The main tool of the proof is a theorem of R. Kaufman from [RK] on the existence of an additive function separating superadditive (m) and subadditive (M) ones, all defined in a commutative semigroup, and such that $m \leq M$.

2.6. Inequalities

One of the main topics of Kominek's research were inequalities. Some of them were already mentioned, but we would like to underline results contained in [11], [22], [26], [30], [33], [36]–[40], [51]–[53], [55], [71], [73] and [74]. Let us look closer to the papers [71] and [73]. The first of these papers was written jointly with Katarzyna Troczka-Pawelec, the youngest of Zygfryd's PhDs. The paper deals with the inequality

$$\varphi(x+y) + \varphi(x-y) \leq 2\varphi(x) + 2\varphi(y), \tag{2.11}$$

where $x, y \in X$ and the authors assume that X is a topological group. We call function φ satisfying (2.11) *subquadratic*. Let us adopt the following definition.

We say that 2-divisible topological group X has the property $(\frac{1}{2})$ if and only if for every neighbourhood V of zero there exists a neighbourhood W of zero such that

$$\frac{1}{2}W \subset W \subset V.$$

The main result reads as follows.

THEOREM 2.26. *Let X be a 2-divisible topological abelian group having the property $(\frac{1}{2})$, which is generated by any neighbourhood of zero in X*[2]*. Assume that a subquadratic function $\varphi\colon X \to \mathbb{R}$ satisfies the following conditions:*

(i) $\varphi(0) \le 0$;

(ii) *φ is locally bounded from below at a point of X;*

(iii) *φ is upper semicontinuous at zero.*

Then φ is continuous everywhere in X.

In the proof of Theorem 2.26 the authors in particular use the following observation.

OBSERVATION 1. Let X be a 2-divisible topological group having the property $(\frac{1}{2})$, $f\colon X \to \mathbb{R}$ be an arbitrary function. Then for every $u \in X$, each $\varepsilon > 0$ and every neighbourhood W_0 of $0 \in X$ there exists a neighbourhood W of $0 \in X$ such that $\frac{1}{2}W \subset W \subset W_0$ and

$$\inf\{f(v); v \in u + W\} + \varepsilon \ge \inf\{f(v'); v' \in u + \frac{1}{2}W\}.$$

The same Observation appears in [73] as Lemma 1. It serves, among others, to prove the main result from the paper.

THEOREM 2.27. *Let $\varphi\colon X \to \mathbb{R}$ be a function satisfying the so called Drygas inequality*

$$\varphi(x+y) + \varphi(x-y) \le 2\varphi(x) + \varphi(y) + \varphi(-y),$$

and

$$\varphi(0) = 0,$$

If φ is locally bounded below at each point of X and upper semicontinuous at zero, then it is continuous in X.

[2]We say that a group X is generated by a set U if $X = \bigcup_{n\in\mathbb{N}} 2^n U$.

Theorem 2.27 implies then a series of other results stating the continuity of φ which is bounded below only at points $x_0 \neq 0$ and $-x_0$ (Theorem 2 in [73]) or is continuous at 0 (Theorem 3 in [73]) or it is semicontinuous at 0 and bounded below at a point $x_0 \neq 0$ (Theorem 4 in [73]).

We cannot refrain from mentioning the unique paper written jointly with our late colleague Zbigniew Gajda (cf. [37]). In that paper they replace commutativity or amenability of a semigroup S by its *weak commutativity* i.e. the condition stated as follows.

Definition 2.28. We say that the semigroup $(S, \cdot)$ is weakly commutative, iff

$$\bigwedge_{x,y\in S} \bigvee_{n\in\mathbb{N}} (xy)^{2^n} = x^{2^n} y^{2^n}.$$

Let $(S, \cdot)$ be a semigroup and consider functions $f, g \colon S \to \mathbb{R}$ satisfying the following conditions.

$$f(xy) \leq f(x) + f(y), \tag{2.12}$$

$$g(xy) \geq g(x) + g(y), \tag{2.13}$$

$$g(x) \leq f(x), \tag{2.14}$$

for every $x, y \in S$. The notion of weak commutativity has been used by several authors but Gajda and Kominek proved in particular the following.

Theorem 2.29. *Suppose that the semigroup $(S, \cdot)$ is weakly commutative and let $f, g \colon S \to \mathbb{R}$ satisfy* (2.12)*,* (2.13) *and* (2.14)*. Moreover, assume that*

$$\sup\{f(x) - g(x) : x \in S\} < \infty. \tag{2.15}$$

Then there exists a unique additive function $a \colon S \to \mathbb{R}$ such that

$$g(x) \leq a(x) \leq f(x), \tag{2.16}$$

for every $x \in S$.

They proved also another theorem using the assumption of weak commutativity, instead of (2.15) assuming that S is a group.

Theorem 2.30. *Let $(S, \cdot)$ be weakly commutative group. If $f, g \colon S \to \mathbb{R}$ satisfy* (2.12)*,* (2.13) *and* (2.14) *then there exists and additive functional $a \colon S \to \mathbb{R}$ such that* (2.16) *holds for every $x \in S$.*

Instead of assuming that S is a group it is possible to admit that S is amenable[3]. The authors of [37] showed the following.

THEOREM 2.31. *Let* $(S,\cdot)$ *be weakly commutative and amenable semigroup. If* $f, g\colon S \to \mathbb{R}$ *satisfy* (2.12), (2.13) *and* (2.14) *then there exists and additive functional* $a\colon S \to \mathbb{R}$ *such that* (2.16) *holds for every* $x \in S$.

2.7. Alienation

In the paper [78] together with Justyna Sikorska, Zygfryd proved a theorem on alienation of the logarithmic and exponential equations. They considered the equation

$$f(xy) - f(x) - f(y) = g(x+y) - g(x)g(y) \tag{2.17}$$

in two cases

A) $x, y \in \mathbb{R}$,

B) $x, y \in \mathbb{R} \setminus \{0\}$.

In particular, they proved the following.

THEOREM 2.32. *Assume that* $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *satisfy* (2.17) *for* $x, y \in \mathbb{R} \setminus \{0\}$. *If* $g(1) \neq 1$ *or* $f(1) \neq 0$, *then*

$$\begin{aligned} &g(x+y) = g(x)g(y), \quad x, y \in \mathbb{R}, \\ \textit{and} \quad &f(xy) = f(x) + f(y), \quad x, y \in \mathbb{R} \setminus \{0\}, \end{aligned} \tag{2.18}$$

or there exist $\alpha \in \mathbb{R} \setminus \{0\}$ *and a function* $F\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *satisfying* $F(xy) = F(x) + F(y)$ *for all* $x, y \in \mathbb{R} \setminus \{0\}$ *such that*

$$\begin{aligned} &g(x) = \alpha x + (1-\alpha), \quad x \in \mathbb{R}, \\ \textit{and} \quad &f(x) = F(x) - \alpha^2 x - \alpha(1-\alpha), \quad x \in \mathbb{R} \setminus \{0\}, \end{aligned} \tag{2.19}$$

or there exist $\beta \in \mathbb{R} \setminus \{1\}$ *and a function* $F\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *satisfying* $F(xy) = F(x) + F(y)$ *for all* $x, y \in \mathbb{R} \setminus \{0\}$ *such that*

$$g(x) = \beta, \quad x \in \mathbb{R}, \quad \textit{and} \quad f(x) = F(x) + \beta^2 - \beta, \quad x \in \mathbb{R} \setminus \{0\}. \tag{2.20}$$

[3] S is amenable if the space $B(S,\mathbb{R})^*$ (conjugate to the space $B(S,\mathbb{R})$ of all bounded on S real functions) contains at least one right- or left invariant mean.

If $g(1) = 1$, $f(1) = 0$ and g is continuous at the origin, then $g(x) = 1, x \in \mathbb{R}$ and $f(xy) = f(x) + f(y), x, y \in \mathbb{R} \setminus \{0\}$. Conversely, each pair of functions described by (2.18), (2.19) *and* (2.20) *is a solution of* (2.17) *for $x, y \in \mathbb{R} \setminus \{0\}$ with any real α and β.*

They asked in [78] the following open question.

Problem 2. Assume $g(1) = 1$ and $f(1) = 0$. Find all solutions of (2.17) for $x, y \in \mathbb{R} \setminus \{0\}$ without assuming the continuity of g at the origin.

The Problem was answered by Gyula Maksa in [GM]. He proved the following

Theorem 2.33. *Let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$. Then* (2.17) *for all $x, y \in \mathbb{R} \setminus \{0\}$ and $g(1) = 1$ and $f(1) = 0$ hold if and only if there are additive functions $a, A\colon \mathbb{R} \to \mathbb{R}$ such that $A(1) = 0$ and*

$$f(x) = a(\ln |x|), \quad x \in \mathbb{R} \setminus \{0\}, \quad \text{and} \quad g(x) = \exp A(x), \quad x \in \mathbb{R}.$$

The result obtained by Maksa is another proof of influence Zygfryd has had on other mathematicians.

Let us note that the list of collaborators of Zygfryd Kominek counts several names: Karol Baron, Bogdan Batko, Zbigniew Gajda, Roman Ger, Eugeniusz Głowacki, Bożena Kominek, Marek Kuczma, Janusz Matkowski, Harry I. Miller, Jacek Mrowiec, Ludwig Reich, Maciej Sablik, Jens Schwaiger, Justyna Sikorska, Jacek Tabor, Katarzyna Troczka-Pawelec, Władysław Wilczyński and Tomasz Zgraja. The Erdös number of Zygfryd is 3.

3. Teacher and friend

Zygfryd Kominek was born on July 30th, 1945 in Radlin, a town south of Katowice. He attended schools there and in 1963 he started his studies in mathematics at the branch of Jagiello University in Katowice. Here he obtained master's degree in 1968, and began his employment after the Summer holidays in 1968 – but it was already the brand new Silesian University. He received his PhD degree in 1974, and his supervisor was Bogdan Choczewski from Cracow. In 1991 he got the habilitation from the Warsaw University of Technology. Since 2008 Zygfryd Kominek is full professor in mathematics.

Professor Kominek has been working in the Silesian University for almost 50 years. In the meantime, he was employed in some other academic institutions: branch of the Łódź University of Technology in Bielsko-Biała (from 1991 till 2001) and then in the University of Bielsko-Biała (from 2001 till 2003). He was employed also in Częstochowa (in the University of Technology from 1992 till 1998, and in the Jan Długosz University from 2004 till 2006). In 2006 he started to work in the Katowice Institute of Information Technologies, he was there till 2011.

At the Institute of Mathematics of the Silesian University, Zygfryd Kominek served as the director (1996–2002) and deputy director (1992–1996 and 2012–2015), he was also a deputy chairman of the Publishing Committee of the Silesian University and a representative of the faculty in the Senate of the University. I have to underline in particular his service as deputy director from 2012 till 2015. He was then responsible for research in the Institute, and I, as the director of the Institute, owe him a lot. Kominek was also the Editor in Chief of Annales Mathematicae Silesianae.

He was leading the prestigious "Tuesday" Seminar on Functional Equations for several years (from October 2009 till June 2015). The seminar was founded in 1963 by Marek Kuczma, and has been continued, meeting every Tuesday of academic year, at 4:15 PM.

Zygfryd is also a teacher. He has five students, chronologically: Tomasz Zgraja, Jacek Mrowiec, Andrzej Olbryś, Michał Lewicki and Katarzyna Troczka-Pawelec. One of them, Andrzej Olbryś, has received his habilitation in 2019. He has joint papers with three of them, and Andrzej Olbryś has been under his influence ever since defending the PhD. His habilitation is actually composed of results touching in general the different kinds of convexity and creatively developing the concepts of "sandwich theorems" and supporting functionals.

Zygfryd took part in numerous conferences on functional equations and inequalities, and real functions, usually giving talks on his own results and presenting remarks and problems that inspired many mathematicians. In the present volume you will find a paper by Harald Fripertinger who answered a Kominek's question asked during the 54th ISFE held in Hajdúszoboszló in Hungary, in 2016 (cf. [RI]).

For many years I participated in the "Monday" seminar on functional equations and inequalities in several variables, founded in 1982 and lead continuously till 2016 by Roman Ger. Since October 2016 I became the leader of the seminar. When I open the meeting, on my right, close to the window, I can see two friends: Roman and Zyga, with whom I wrote the paper [19]. They are still attending the meetings despite the fact that for some time they are both emeriti. Still, their comments and remarks are extremely valuable and I am disappointed indeed, if for some reasons one of them is missing. It is hard for me to imagine the seminar without both of them. We organized in 2015 a special conference, named Ger–Kominek Workshop in Mathematical Analysis and Real Functions, dedicated to the two friends on the occasion of their seventieth anniversaries. More than a hundred of mathematicians came from all over the world and congratulated our colleagues. Though the celebration

Figure 1. Prof. Zygfryd Kominek and three of his PhD students: Dr. Tomasz Zgraja, Prof. Andrzej Olbryś, Dr. Jacek Mrowiec during the Ger–Kominek Workshop on Analysis and Real Functions, Katowice 2015 (Photograph by Ewa Kurzeja)

was great, it did not mean the crowning of their mathematical activity, which has gained significant momentum at that time.

We wish you, Dear Zygfryd, many successes in future, many new results, and many meetings with friends!

Zygfryd Kominek: List of publications

[1] *On the sum and difference of two sets in topological vector spaces*, Fund. Math. **71** (1971), no. 2, 165–169.

[2] *The Galton-Watson process with immigration*, (Polish), in: M. Kuczma (ed.), *Functional Equations in the Theory of Stochastic Processes* (Polish), Publ. Silesian Univ. Katowice, No. **47**, Wydawn. Uniw. Śląsk., Katowice, 1972, pp. 203–210.

[3] *Some generalization of the theorem of S. Piccard*, Zeszyty Naukowe UŚl. **4** (1973), 31–33.

[4] *On the existence of a convex solution of the functional equation* $\varphi(x) = h(x, \varphi[f(x)])$, Ann. Polon. Math. **30** (1974), 1–4 (with J. Matkowski).

[5] C^r*-solutions of a system of functional equations*, Ann. Polon. Math. **30** (1974), 191–203.

[6] *On the uniqueness and the existence of solutions with some asymptotic properties of a system of functional equations*, Prace Mat. UŚl. **7** (1977), 57–63.
[7] *On some classes connected with the continuity of additive and Q-convex functions*, Prace Mat. UŚl. **8** (1978), 60–63 (with B. Kominek).
[8] *On the behaviour of C^r-solutions of a system of functional equations*, Revue Rom. Math. Pures et Appl. **23** (1978), no. 2, 1067–1076.
[9] *Some theorems on differentiable solutions of a system of functional equations of n-th order*, Ann. Polon. Math. **37** (1980), no. 1, 71–91.
[10] *On the functional equation $\varphi(x) = h(x, \varphi[f(x)])$*, Demonstr. Math. **14** (1981), no. 4, 1031–1045.
[11] *On the continuity of Q-convex functions and additive functions*, Aequationes Math. **23** (1981), no. 2–3, 146–150.
[12] *A generalization of K. Goebel's and J. Matkowski's theorems*, Prace Mat. UŚl. **12** (1982), 30–33.
[13] *Some remarks on the Boyd-Wong theorem*, Glas. Mat. Ser. III **17(37)** (1982), no. 2, 313–319.
[14] *Some remarks on the set-classes A_C^N and B_C^N*, Comment. Math. **23** (1983), no. 1, 49–52.
[15] *Measure, category and the sum of sets*, Amer. Math. Monthly **90** (1983), no. 8, 561–562.
[16] *On a decomposition of the space of real numbers*, Glas. Mat. Ser. III **19(39)** (1984), no. 2, 231–233.
[17] *The recurrent sequences of inequalities*, Prace Mat. UŚl., Ann. Math. Sil. **3(15)** (1990), 41–44.
[18] *Some remarks on a theorem of Steinhaus*, Glasnik Mat. Ser. III **20(40)** (1985), no. 2, 337–344 (with H.I. Miller).
[19] *Generalized Smítal's lemma and a theorem of Steinhaus*, Rad. Mat. **1** (1985), no. 1, 101–119 (with R. Ger and M. Sablik).
[20] *On the functional equations $\varphi(x) = a\varphi(ax) + (1-a)\varphi(1-(1-a)x)$*, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.) **30(78)** (1986), no. 4, 327–334 (with J. Matkowski).
[21] *On some property of J-concave functions*, Rad. Mat. **3** (1987), no. 1, 143–147.
[22] *On additive and convex functionals*, Rad. Mat. **3** (1987), no. 2, 267–279.
[23] *On an equivalent form of a Steinhaus theorem*, Mathematica (Cluj) **30(53)** (1988), no. 1, 25–27.
[24] *On a problem of K. Nikodem*, Arch. Math. (Basel) **50** (1988), no. 3, 287–288.
[25] *Boundedness and continuity of additive and convex functionals*, Aequationes Math. **37** (1989), no. 2–3 , 252–258 (with R. Ger).
[26] *Theorems of Bernstein-Doetsch, Piccard and Mehdi and semilinear topology*, Arch. Math. (Basel) **52** (1989), no. 6, 595–602 (with M. Kuczma).
[27] *A characterization of convex functions in linear spaces*, Opuscula Math. **5** (1989), 71–74.
[28] *Some properties of decompositions of a commutative group*, Comment. Math. **28** (1989), no. 2, 249–252.
[29] *On a local stability of the Jensen functional equation*, Demonstr. Math. **22** (1989), no. 2, 499–507.
[30] *On the lower hull of convex functions*, Aequationes Math. **38** (1989), no. 2–3, 192–210 (with M. Kuczma).
[31] *Convex functions in linear spaces*, Prace Mat. UŚl. **1087** (1989), 70 pp.
[32] *On functions that preserve a certain class of sets*, Boll. Un. Mat. Ital. A (7) **4** (1990), no. 2, 165–172 (with H.I. Miller).
[33] *Additive and J-convex functions majorized by measurable functions*, Rad. Mat. **6** (1990), no. 1, 49–53.
[34] *Convex functions and epigraphs*, Rad. Mat. **6** (1990), no. 1, 97–110.
[35] *On approximately subadditive functions*, Demonstr. Math. **23** (1990), no. 1, 155–160.

[36] *p-convex functions in linear spaces*, Ann. Polon. Math. **53** (1991), no. 2, 91–108 (with M. Kuczma).
[37] *On separation theorems for subadditive and superadditive functionals*, Studia Math. **100** (1991), no. 1, 25–38 (with Z. Gajda).
[38] *On (a, b)-convex functions*, Arch. Math. (Basel) **58** (1992), no. 1, 64–69.
[39] *Theorem of Bernstein-Doetsch in Baire spaces*, Ann. Math. Sil. **5** (1991), 10–17 (with M. Kuczma).
[40] *Convex functions and some set classes*, Rocznik Nauk.-Dydakt. WSP w Krakowie, Prace Mat. **13** (1993), 187–196.
[41] *Note on polynomial functions*, Ann. Math. Sil. **7** (1993), 7–15.
[42] *On a result of W. Stadje*, Zeszyty Nauk. Politech. Śląsk. w Gliwicach, Mat.-Fiz. z. **68** (1993), 153–163.
[43] *Boundedness and linearity of additive functionals*, Opuscula Math. **14** (1994), 83–87.
[44] *On stability of the Pexider equation on semigroups*, in: Th.M. Rassias and J. Tabor (eds.), *Stability of Mappings of Hyers-Ulam Type*, Hadronic Press, Palm Harbor, 1994, pp. 111–116 (with E. Głowacki).
[45] *On stability of the homogeneity condition*, Results Math. **27** (1995), no. 3–4, 373–380 (with J. Matkowski).
[46] *Translations in normed spaces*, Publ. Math. Debrecen **49** (1996), no. 3–4, 295–299.
[47] *Note on selection of Jensen set-valued functions*, Tatra Mt. Math. Publ. **14** (1998), 75–79.
[48] *On additive functions fulfilling some additional condition*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **207** (1998), 35–42 (with L. Reich and J. Schwaiger).
[49] *On sets for which the difference set is the whole space*, Rocznik Nauk.-Dydakt. WSP w Krakowie, Prace Mat. **16** (1999), 45–51 (with W. Wilczyński).
[50] *On some properties of the quasisymmetry quotients of functions*, Aequationes Math. **58** (1999), no. 1–2, 93–99.
[51] *Convex functions with respect to logarithmic mean and sandwich theorem*, Acta Univ. Carolin. Math. Phys. **40** (1999), no. 2, 75–78 (with T. Zgraja).
[52] *Generalized norms and convexity*, Publ. Math. Debrecen **60** (2002), no. 1–2, 63–73 (with B. Batko and J. Tabor).
[53] *Solution of Rolewicz's problem*, http://www.siam.org/journals/problems/01-005.htm (with B. Choczewski and R. Girgensohn).
[54] *On a problem of Wu Wei Chao*, Ann. Math. Sil. **16** (2002), 17–20.
[55] *A proof of S.Rolewicz's conjecture*, Ann. Acad. Pedagog. Crac. Stud. Math. **2** (2002), folia 13, 5–11 (with B. Choczewski).
[56] *A continuity result on t-Wright-convex functions*, Publ. Math. Debrecen **63** (2003), no. 1–2, 213–219.
[57] *On functionals with the Cauchy difference bounded by a homogeneous functional*, Bull. Polish Acad. Sci. Math. **51** (2003), no. 3, 301–307 (with K. Baron).
[58] *On Hyers-Ulam stability of the Pexider equation*, Demonstr. Math. **37** (2004), no. 2, 373–376.
[59] *On a functional equation arising in the theory of quasiconformal mappings*, Aequationes Math. **70** (2005), no. 1–2, 37–42.
[60] *A few remarks on almost C-polynomial functions*, Math. Slovaca **55** (2005), no. 5, 555–561.
[61] *Nonstability results in the theory of convex functions*, C. R. Math. Acad. Sci. Soc. R. Can. **28** (2006), no. 1, 17–23 (with J. Mrowiec).
[62] *Some remarks on subquadratic functions*, Demonstr. Math. **39** (2006), no. 4, 751–758 (with K. Troczka).
[63] *On a problem of Th. Davison*, Tatra Mt. Math. Publ. **34** (2006), part II, 229–235.
[64] *Remarks on the stability of some quadratic functional equations*, Opuscula Math. **28** (2008), no. 4, 517–527.
[65] *Stability of a quadratic functional equation on semigroups*, Publ. Math. Debrecen **75** (2009), no. 1–2, 173–178.

[66] *On pseudoadditive mappings*, Ann. Math. Sil. **22** (2008), 41–44.
[67] *On a Jensen-Hosszú equation I*, Ann. Math. Sil. **23** (2009), 57–60.
[68] *On non-measurable subsets of the real line*, in: M. Filipczak and E. Wagner-Bojakowska (eds.), *Real functions, density topology and related topics*, Dedicated to Profesor Władysław Wilczyński, Łódź University Press, Łódź, 2011, pp. 63–66.
[69] *An interplay between Jensen's and Pexider's functional equations on semigroups*, Ann. Univ. Sci. Budapest. Sect. Comput. **35** (2011), 107–124 (with R. Ger).
[70] *On a Jensen-Hosszú equation, II*, Math. Inequal. Appl. **15** (2012), no. 1, 61–67 (with J. Sikorska) .
[71] *Continuity of real valued subquadratic functions*, Comment. Math. **51** (2011), no. 1, 71–75 (with K. Troczka).
[72] *On the Hyers-Ulam stability of Pexider-type extension of the Jensen-Hosszú equation*, Bull. Int. Math. Virtual Inst. **1** (2011), 53–57.
[73] *On a Drygas inequality*, Tatra Mt. Math. Publ. **52** (2012), 65–70.
[74] $(C-K)$*-strongly* n*-convex functions in linear spaces*, Comment. Math. **53** (2013), no. 2, 189–196.
[75] *Fix-point free affine transformations having invariant lines*, Ann. Univ. Sci. Budapest Sect. Comput. **41** (2013), 37–42.
[76] *Stability aspects of the Jensen-Hosszú equation*, in: M. FIlipczak and E, Wagner-Bojakowska, *Traditional and present day topics in real analysis*, Dedicated to Professor Jan Stanisław Lipiński on the occasion of his 90th birthday, University of Łódź, Łódź, 2013, pp. 307–324.
[77] *On pexiderized Jensen-Hosszú functional equation on the unit interval*, J. Math. Anal. Appl. **409** (2014), no. 2, 722–728.
[78] *Alienation of the logarithmic and exponential functional equations*, Aequationes Math. **90** (2016), no. 1, 107–121 (with J. Sikorska).

References

[MR1] MR791750 (87e:28008)
[MR2] MR986915 (90c:28001)
[MR3] MR1037927 (91c:39010)
[RA] *Modern Real Analysis*, editors: J. Hejduk, S. Kowalczyk, R. J. Pawlak, and M. Turowska, Dedicated to Professors Roman Ger, Jacek Jędrzejewski, Zygfryd Kominek, Wydawnictwo Uniwersytetu Łódzkiego, Łódź, 2015.
[AA] A. Abian, *The outer and inner measure of a non-measurable set*, Boll. Un. Mat. Ital. (4) **3** (1970), 555–558.
[BSV] K. Baron, M. Sablik, and P. Volkmann, *On decent solutions of a functional congruence*, Rocznik Nauk.-Dydakt. Akad. Pedagog. w Krakowie, Prace Mat. **17** (2000), 27–40.
[GK] R. Ger, M. Kuczma, *On the boundedness and continuity of convex functions and additive functions*, Aequationes Math. **4** (1970), 157–162.
[DH] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.
[HU] D.H. Hyers, S. Ulam, *On approximate isometries*, Bull. Amer. Math. Soc. **51** (1945), 288–292.
[RK] R. Kaufman, *Interpolation of additive functionals*, Studia Math. **27** (1966), 269–272.
[MEK] M.E. Kuczma, *On discontinuous additive functions*, Fund. Math. **66** (1970), 383–392.

[KK] M.E. Kuczma and M. Kuczma, *An elementary proof and an extension of a theorem of Steinhaus*, Glasnik Mat. Ser. III **6(26)** (1971), 11–18.

[MK] M. Kuczma, *On some set classes occurring in the theory of convex functions*, Comment. Math. **17** (1973), 127–135.

[KS] M. Kuczma and J. Smítal, *On measures connected with the Cauchy equation*, Aequationes Math. **14** (1976), 421–428.

[NK] N. Kuhn, *A note on t-convex functions*, in: W. Walter (ed.), *General Inequalities, 4*, Oberwolfach, 1983, Internat. Ser. Numer. Math., vol. **71**, Birkhäuser Verlag, Basel-Boston, 1984, pp. 269–276.

[GM] Gy. Maksa, *On the alienation of the logarithmic and exponential Cauchy equations*, Aequationes Math. **92** (2018), 543–547.

[HM] H.I. Miller, *An incompatibility result*, Rev. Roumaine Math. Pures Appl. **26** (1981), 1217–1220.

[NG] C.T. Ng, *Functions generating Schur-convex sums*, in: W. Walter (ed.), *General Inequalities, 5*, Oberwolfach, 1986, Internat. Ser. Numer. Math., vol. **80**, Birkhäuser Verlag, Basel-Boston, 1987, pp. 433–438.

[AO] A. Olbryś, *A support theorem for generalized convexity and its applications*, J. Math. Anal. Appl. **458** (2018), 1044–1058.

[OC] W. Orlicz and Z. Ciesielski, *Some remarks on the convergence of functionals on bases*, Studia Math. **16** (1958), 335–352.

[SP] S. Piccard, *Sur les ensembles parfaits*, Mém. Univ. Neuchâtel, vol. **16**, Secrétariat de l'Université, Neuchâtel, 1942, 172 pp.

[RI] *Report of Meeting*, Aequationes Math. **91** (2017), 1157–1204.

[GR] G. Rodé, *Eine abstrakte Version des Satzes von Hahn-Banach*, Arch. Math. (Basel) **31** (1978), 474–481.

[WS1] W. Sander, *Verallgemeinerungen eines Satzes von S. Piccard*, Manuscripta Math. **16** (1975), 11–25.

[WS2] W. Sander, *Verallgemeinerungen eines Satzes von H. Steinhaus*, Manuscripta Math. **18** (1976), 25–42.

[FS] F. Skof, *Local properties and approximation of operators*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.

[JT] J. Tabor, *Hosszú functional equation on the unit interval is not stable*, Publ. Math. Debrecen **49** (1996), 335–340.

[WW] W. Wilczyński, *Theorems of H. Steinhaus, S. Piccard and J. Smítal*, Lecture during the Ger-Kominek Workshop in Mathematical Analysis and Real Functions. Katowice, Silesian University, November 20–21, 2015.

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