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AN ITERATION PROOF OF THE MAXIMUM PRINCIPLE

1. Introduction

We give a nonstandard proof of the Maximum Principle for linear elliptic partial differential equations of the second order. Our intention is to present a new method for the proofs of similar theorems, introduced by N.D. Alikakos [1] and T. Dłotko [2] for the studies of semi-linear partial differential equations of parabolic type.

2. Preliminaries

We will deal with the elliptic equation in divergence form:

$$(1)\qquad \sum_{i,j=1}^{n}(a_{ij}(x)u_{x_j})_{x_i} + \sum_{j=1}^{n} b_j(x)u_{x_j} + c(x)u + f(x) = 0$$

considered in a bounded domain $\Omega\subset R^n$, with suitably smooth boundary. It is assumed throughout the paper that the function u satisfying (1) belongs to $C^2(\Omega)\cap C^0(\bar{\Omega})$. The partial derivatives of the function u are denoted by u_{x_i}, $u_{x_ix_j}$ and from now on all unspecified sums are to be taken from 1 to n. The following properties of the coefficients are globally assumed:

(i) the functions a_{ij} and b_j belong to $C^1(\Omega)$ and for every $x\in\Omega$ and $\xi\in R^n$

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq 0,$$

(ii) the function f is globally bounded in Ω.

The following elementary lemma will be needed in the sequel:

Lemma 1. Let $G_m(x) = (1+x^m)[1-x(1+x^m)^{-\frac{1}{m}}]$ where m is an integer and x is a nonnegative real number. Then

$$G_m(x) \geq \frac{1}{m}. \tag{2}$$

Proof. It is easy to verify the following sequence of estimates:

$$(1+x^m)\left[1-x(1+x^m)^{-\frac{1}{m}}\right] = (1+x^m)^{1-\frac{1}{m}}\left[(1+x^m)^{\frac{1}{m}} - x\right] =$$

$$= (1+x^m)^{1-\frac{1}{m}} \frac{1+x^m-x^m}{(1+x^m)^{1-\frac{1}{m}}+(1+x^m)^{1-\frac{2}{m}}x+ \dots +(1+x^m)^{\frac{1}{m}}x^{m-2}+x^{m-1}} =$$

$$= \frac{1}{1+(1+x^m)^{-\frac{1}{m}}x + \dots + (1+x^m)^{\frac{1-m}{m}}x^{m-1}} \geq$$

$$\geq \frac{1}{\underbrace{1+1+\dots+1}_{m}} = \frac{1}{m}$$

from which it is clear that (2) is satisfied.

3. Main theorem

Theorem 1. If there exists a constant $h>0$ such that $c(x) \leq -h$ for every $x \in \Omega$ and a function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies (1), then

$$\|u\|_{L^\infty(\Omega)} \leq \max\left(m, \frac{M}{h}\right) \tag{3}$$

where $m = \sup_{x \in \partial\Omega} |u(x)|$ and $M = \|f\|_{L^\infty(\Omega)}$.

Proof. We define sets $\Omega_l = \{ x \in \Omega : \rho(x, \partial\Omega) > \frac{1}{l} \}$, where l is an integer and $\rho(x,\partial\Omega)$ denotes the distance from x to $\partial\Omega$. Let us fix the number l. Since the functions a_{ij}, b_j belong to $C^1(\bar{\Omega}_l)$, then in particular for some constant $B_l>0$ we have

$$\left|\sum_j (b_j(x))_{x_j}\right| \leq B_l \quad \text{for } x \in \bar{\Omega}_l. \tag{4}$$

Multiplying (1) by u^{2^k-1} (the number $k\in N$, sufficiently large, is fixed until the limit passage at the end of the proof of Th.1) and integrating the result over Ω_1, we obtain

$$(5)\quad \int_{\Omega_1} \sum_{i,j} (a_{ij}u_{x_j})_{x_i} u^{2^k-1}dx + \int_{\Omega_1} \sum_j b_j u_{x_j} u^{2^k-1}dx +$$
$$+\int_{\Omega_1} cu^{2^k}dx + \int_{\Omega_1} fu^{2^k-1}dx = 0.$$

The first and second components of (5) are integrated by parts and fourth is estimated using the Hölder inequality (with $p=2^k(2^k-1)^{-1}$, $q=2^k$). We obtain the following estimate:

$$(6)\quad \int_{\partial\Omega_1} \sum_{i,j} a_{ij}u_{x_j}\cos(n,x_i)u^{2^k-1}ds -$$
$$-(2^k-1)\int_{\Omega_1} \sum_{i,j} a_{ij}u_{x_j}u_{x_i}u^{2^k-2}dx + 2^{-k}\int_{\partial\Omega_1}\sum_j b_j\cos(n,x_j)u^{2^k}ds -$$
$$- 2^{-k}\int_{\Omega_1}\sum_j (b_j)_{x_j}u^{2^k}dx + \int_{\Omega_1} cu^{2^k}dx +$$
$$+ \left(\int_{\Omega_1} |f|^{2^k}dx\right)^{2^{-k}}\left(\int_{\Omega_1} u^{2^k}dx\right)^{1-2^{-k}} \geq 0.$$

Let us denote by $|\Omega_1|$ the Lebesgue measure of Ω_1 and let $m_1 := \sup_{x\in\partial\Omega_1} |u(x)|$, $H_1 := \int_{\partial\Omega_1}\left|\sum_{i,j} a_{ij}u_{x_j}\cos(n,x_i)\right|ds$ and $H_2 := \int_{\partial\Omega_1}\left|\sum (b_j)_{x_j}\cos(n,x_j)\right|ds$. Since from (i) $\sum_{i,j} a_{ij}u_{x_i}u_{x_j}\geq 0$, then multiplying both sides of (6) by 2^k we conclude that

$$(7)\quad 2^k m_1^{2^k-1}H_1 + m_1^{2^k}H_2 + \int_{\Omega_1} (B_1-2^k h)u^{2^k}dx +$$
$$+ 2^k M|\Omega_1|^{2^{-k}}\left(\int_{\Omega_1} u^{2^k}dx\right)^{1-2^{-k}} \geq 0.$$

This inequality generates the estimate of the quantity $y_k:=\int_{\Omega_1} u^{2^k}dx$. For convenience let us introduce the following notation:

$$\alpha_k:=2^k h-B_1,\ \beta_k:=2^k M|\Omega_1|^{2^{-k}} \text{ and } \gamma_k:=2^k m_1^{2^k-1}H_1+m_1^{2^k}H_2.$$

The inequality (7) then takes the form

$$\gamma_k - \alpha_k y_k + \beta_k y_k^{1-2^{-k}} \geq 0$$

or equivalently

(8) $$\alpha_k y_k - \beta_k y_k^{1-2^{-k}} \leq \gamma_k$$

where $y_k \geq 0$ for every k and α_k, β_k, γ_k are nonnegative for sufficiently large k.

Defining the function F as follows

$$F(y) := y(\alpha_k - \beta_k y^{-2^{-k}}),$$

it is easy to see that for $y \in I:=\left[\left(\frac{\beta_k}{\alpha_k}\right)^{2^k},\infty\right)$ the function F is increasing. The inequality (8) may be rewritten as

(9) $$F(y_k) \leq \gamma_k.$$

Let us define $y^*:=\left(\frac{\beta_k}{\alpha_k}\right)^{2^k} + \alpha_k\gamma_k \in I$. Our aim now is to show that

(9′) $$F(y^*) \geq \gamma_k$$

which, in the presence of (9), implies that

(10) $$y_k \leq y^*.$$

Let us note that: either $\gamma_k=0$, in which case $F(y^*)=0\geq\gamma_k$, or if not, then using Lemma 1 and denoting

$K := \beta_k\left(\alpha_k^{1+2^{-k}}\gamma_k^{2^{-k}}\right)^{-1}$, we find that

$$F(y^*) = \alpha_k^2\gamma_k\left(1+K^{2^k}\right)\left[1 - \frac{K}{\left(1+K^{2^k}\right)^{2^{-k}}}\right] = \alpha_k^2\gamma_k G_{2^k}(K) \geq$$

$$\geq \alpha_k^2\gamma_k 2^{-k} \geq \gamma_k,$$

since $2^{-k}\alpha_k^2 = 2^{-k}(2^k h-B_1)^2 \geq 1$ for sufficiently large k. Thus in both cases we arrive at the inequality (9′).

Next, as a consequence of (10) (for explicit y^*)

$$y_k^{2^{-k}} \le \left[\left(\frac{\beta_k}{\alpha_k}\right)^{2^k} + \alpha_k \gamma_k\right]^{2^{-k}}$$

or using the previous notation

(11) $$\|u\|_{L^{2^k}(\Omega_1)} \le \left[(2^k h - B_1) m_1^{2^k - 1}(2^k H_1 + m_1 H_2) + \left(\frac{2^k M |\Omega_1|^{2^{-k}}}{2^k h - B_1}\right)^{2^k}\right]^{2^{-k}} .$$

It is known ([8], I.3. th.1) that for k tending to infinity $\|u\|_{L^{2^k}(\Omega_1)} \longrightarrow \|u\|_{L^\infty(\Omega_1)}$ and furthermore easy calculation shows that the right-hand side of (11) converges to $\max\left(m_1, \frac{M}{h}\right)$. Thus from (11) we conclude that

(12) $$\|u\|_{L^\infty(\Omega_1)} \le \max\left(m_1, \frac{M}{h}\right).$$

If l tends to infinity, then $\|u\|_{L^\infty(\Omega_1)}$ converges to $\|u\|_{L^\infty(\Omega)}$ and $\sup_{x\in\partial\Omega_1} |u(x)| = m_1$ converges to $m = \sup_{x\in\partial\Omega} |u(x)|$ and then from inequality (12) we obtain the estimate (3), which completes the proof.

Remark 1. Theorem 1 for $f\equiv 0$ coincides with the classical form of the Weak Maximum Principle (see [3], [4], [6], [7]).

4. Consequences of Theorem 1

Other variants of the Maximum Principle will be obtained as the conclusions of Theorem 1. These results are formulated in the following Theorems 2 and 3.

Theorem 2. Let us suppose that the function c is continuous and negative in the set Ω and $f\equiv 0$. If u is a $C^2(\Omega)\cap C^0(\bar{\Omega})$ solution of (1), then

(13) $$\|u\|_{L^\infty(\Omega)} \le \sup_{x\in\partial\Omega} |u(x)|.$$

Proof. If we define the sets Ω_1 as in the previous Theorem and fix an integer l, then there exists a constant $h_1>0$ such that $c(x)\le -h_1<0$ for $x\in\Omega_1$. Using Theorem 1 with $f\equiv 0$, for the function u we obtain the following inequality

(14) $$\|u\|_{L^\infty(\Omega_1)} \le \sup_{x\in\partial\Omega_1} |u(x)|.$$

If l tends to infinity, then from the continuity of the function u, the estimate (13) follows from (14) and the proof is completed.

The assumptions concerning the coefficient c may be weakened still further provided the properties of a_{ij} are improved.

Theorem 3. Let us suppose that the function c is nonpositive in Ω, $f\equiv 0$ and for every $x\in\Omega$ and for all $\xi\in R^n$ with $\xi\neq 0$

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j>0.$$

If a function $u\in C^2(\Omega)\cap C^0(\bar\Omega)$ satisfies (1), then the estimate (13) holds good.

Proof. Let $\lambda(x)$ denote smallest eigenvalue of the matrix $[a_{ij}(x)]_{i,j}$. Then $\lambda(x)>0$ and

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j\geq\lambda(x)\sum_j \xi_j^2 \quad \text{for} \quad x\in\Omega$$

and from (i) it follows that the function λ is continuous.

Let us define the sets Ω_l as in Theorem 1 and fix an integer l. Then since $\lambda\in C^0(\bar\Omega_l)$, there exists a constant $\varepsilon_l>0$ such that $\lambda(x)\geq\varepsilon_l$ for all $x\in\bar\Omega_l$.

Since the set Ω is bounded, then there exist positive constants r, d_1, d_2 such that for every $x=(x_1,\dots,x_n)\in\Omega$ the condition $d_1\leq x_i+r\leq d_2$ for $i=1,\dots,n$ holds good. Let us introduce a function $v:\Omega_l\longrightarrow R$ with the following equality:

$$u(x) = [1-\exp(-s(x))]v(x) \tag{15}$$

where $s(x)=\sum_k s_k(x_k+r)$ and the positive constants s_k will be chosen later (the similar function was used in [6] p. 146). Replacing u(x) by $[1-\exp(-s(x))]v(x)$ in equation (1) we obtain the equation for the function v:

$$\sum_{i,j}(a'_{ij}v_{x_j})_{x_i} + \sum_i b'_i v_{x_i} + c' v = 0. \tag{16}$$

Here $a'_{ij}(x) = [1-\exp(-s(x))]a_{ij}(x)$,

$$b'_i(x) = \sum_j a_{ij}(x) s_j \exp(-s(x)) + b_i(x)[1-\exp(-s(x))]$$

and

$$c'(x) = \exp(-s(x))\Big\{\sum_j \Big[\sum_i (a_{ij}(x))_{x_i} + b_j(x)\Big] s_j - \sum_{i,j} a_{ij} s_i s_j\Big\} +$$

$$+ c(x)[1-\exp(-s(x))].$$

Since $c(x) \le 0$, then also

(17) $$c(x)[1-\exp(-s(x))] \le 0.$$

Moreover,

(18) $$\sum_j \Big[\sum_i (a_{ij})_{x_i} + b_j\Big] s_j - \sum_{i,j} a_{ij} s_i s_j \le \sum_j A|s_j| - \varepsilon_1 \sum_j s_j^2$$

where the constant A is such that $\Big|\sum_i (a_{ij})_{x_i} + b_j\Big| \le A$ for $j=1,\dots,n$.

Then for all sufficiently large constants s_k we have

(19) $$\sum_j A|s_j| - \varepsilon_1 \sum_j s_j^2 \le -h_1 < 0$$

where h_1 is a constant. From (17), (18) and (19) it follows that $c'(x) \le -h_1 \exp\big(-\sum_k s_k d_2\big)$. Applying Theorem 1 to equation (16) we obtain the inequality

$$\|v\|_{L^\infty(\Omega_1)} \le \sup_{x\in\partial\Omega_1} |v(x)|$$

which for u found from (15) takes the form

(20) $$\left\|\frac{u(x)}{1-\exp(-s(x))}\right\|_{L^\infty(\Omega_1)} \le \sup_{x\in\partial\Omega_1} \left|\frac{u(x)}{1-\exp(-s(x))}\right|.$$

Since $d_1 \le x_k + r \le d_2$, then from (20) we find that

(21) $$\|u\|_{L^\infty(\Omega_1)} \le \frac{1-\exp\big(-d_2 \sum_k s_k\big)}{1-\exp\big(-d_1 \sum_k s_k\big)} \sup_{x\in\partial\Omega_1} |u(x)|.$$

The constants s_k may be chosen arbitrarily large, thus from (21) it follows that

$$\|u\|_{L^\infty(\Omega_1)} \le \sup_{x\in\partial\Omega_1} |u(x)|.$$

After the limit passage with l to infinity we obtain the required inequality (13) thus completing the proof.

5. Final remarks

The object of the present paper was to obtain new proofs of known facts. These proofs, based on the iterative estimation technique, are different from preceding proofs (compare [3], [4], [6], [7]). The iteration technique, in contrast to classical methods, may be used to study weak solutions of the elliptic equation (see [2], [4], [5]).

It is noteworthy that the assumption (i) admits the equality $a_{ij}\equiv 0$ for i,j=1,...,n and then Theorem 1 covers the case of linear equation of the first order and confirms the Maximum Principle for this equation.

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