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Tomasz Dłotko

AN L^p APPROACH TO A PARABOLIC PROBLEM WITH BLOWING UP SOLUTIONS

1. Introduction

We want to examine the simple exemplary nonlinear parabolic equation

$$(1)\qquad u_t = \Delta u + \lambda |u|^{q-1} u, \quad \lambda > 0,\ q > 1,$$

considered with the Dirichlet or Neumann type boundary conditions:

$$(2)\qquad u(0,x) = u_0(x), \quad x \in \Omega \subset R^n,$$

$$(3')\qquad u(t,x) = 0 \quad \text{for} \quad x \in \partial\Omega, \quad \text{or}$$

$$(3'')\qquad \frac{\partial u(t,x)}{\partial n} = 0 \quad \text{for} \quad x \in \partial\Omega,$$

where Ω is a bounded smooth ($\partial\Omega \in C^{2+\beta}$ with some $\beta \in (0,1)$) domain, $t \geq 0$ and n denotes the inward normal vector to $\partial\Omega$. If the specification of the boundary condition is not necessary we refer simply to (3). With the additional assumption $u_0(x) \geq 0$, our considerations remain valid for nonnegative solutions of the equation

$$(4)\qquad u_t = \Delta u + \lambda u^q,$$

subjected to the above initial-boundary conditions, since for nonnegative solutions problems (1)-(3) and (4), (2), (3) coincide. The blow up phenomenon (the solution may cease to exist, becoming unbounded in a finite time T) for solutions of the presented problems has recently been extensively studied (e.g. [2], [6], [7], [12]). Our purpose here is to describe the behaviour of solutions of (1)-(3) in depend of u_0, λ and q, basing on the $L^p(\Omega)$ estimates originated by N.D.Alikakos [1] and used recently in [3], [4]. We propose also the proof of local existence of the Hölder continuous solutions of (1)-(3).

The standard notation of the Sobolev spaces [9], [10] and Holder spaces [5], [10] is used throughout the paper.

2. The proof of existence

A simple proof of local existence of the uniformly Hölder continuous solution of (1)-(3) is presented first. In particular the maximal time T of existence of the solution is estimated from below (compare [12]).

Theorem 1. For arbitrary initial function $u_0 \in C^{2+\beta}(\bar{\Omega})$ satisfying suitable compatibility conditions ($u_0 = 0$ and $\Delta u_0 + |u_0|^{q-1} u_0 = 0$ for $x \in \partial\Omega$ in the case of the Dirichlet condition (3′) or $\frac{\partial u_0}{\partial n} = 0$ on $\partial\Omega$ for the Neumann condition (3″)) there exists a unique $C^{1+\frac{\alpha}{4},\, 2+\frac{\alpha}{2}}([0,t_1-\varepsilon]\times\bar{\Omega})$ solution u of (1)-(3) ($\alpha = \min\{\beta; \frac{1}{2}\}$, $\varepsilon \in (0,t_1)$ arbitrary). The estimate of the existence time T; $t_1 \leq T \leq +\infty$ follows from the formula (9).

Proof. Uniqueness. Let u_1 and u_2 be two different solutions of (1)-(3) subjected to the same initial function u_0, both existing for $t \in [0,\tau)$. The difference $U = u_1 - u_2$ satisfies

$$U_t = \Delta U + \lambda(|u_1|^{q-1} u_1 - |u_2|^{q-1} u_2),$$

and since the function $f(z) = |z|^{q-1} z$ is increasing and differentiable, with $f'(z) = q|z|^{q-1}$, and the solutions

u_1, u_2 are bounded; $|u_1|, |u_2| \leq M$ for $t \in [0, \tau - \varepsilon']$, $x \in \bar{\Omega}$ ($\varepsilon' > 0$ small), we verify that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega U^2(t,x)dx \leq - \int_\Omega \sum_{i=1}^{n} (U_{x_i}(t,x))^2 dx +$$

$$+ \lambda q \int_\Omega |\tilde{u}|^{q-1} U^2(t,x)dx \leq \lambda q M^{q-1} \int_\Omega U^2(t,x)\, dx.$$

The last estimate ensures that $U(t,x) = 0$ for $t \in [0, \tau - \varepsilon']$, $x \in \bar{\Omega}$, since $U(0,x)$ was equal to zero.

We proceed to derive an a priori estimate of the $L^\infty(\Omega)$ norm of $u(t,\cdot)$. Multiplying (1) by u^{2k-1}; $k \in N$, integrating over Ω and by parts, we get:

(5)
$$\frac{1}{2k}\frac{d}{dt}\int_\Omega u^{2k}(t,x)\, dx =$$

$$= - \frac{2k-1}{k^2}\int_\Omega \sum_{i=1}^{n} ((u^k)_{x_i})^2\, dx + \lambda \int_\Omega |u|^{2k+q-1}\, dx.$$

Let k be now taken so large that $4k-n(q-1) > 0$. Consider the version of the Nirenberg-Gagliardo estimate [8]

(6)
$$\exists_{c>0} \forall_v \|v\|_{L^{2+\frac{q-1}{k}}(\Omega)} \leq c\|v\|^{\theta}_{H^1(\Omega)}\|v\|^{1-\theta}_{L^2(\Omega)},$$

valid for every $v \in H^1(\Omega)$ with $\theta = \frac{n(q-1)}{2(2k+1-1)}$. If $v \in H^1_0(\Omega)$, then the $H^1(\Omega)$ norm in (6) will be replaced by the $L^2(\Omega)$ norm of the space gradient $\nabla_x v = (v_{x_1}, \dots, v_{x_n})$. Using the Young inequality (where $a,b \geq 0$, $\bar{\varepsilon} > 0$, $m > 1$);

$$ab \leq \frac{1}{m}(\bar{\varepsilon} a)^m + \frac{m-1}{m}(\bar{\varepsilon}^{-1} b)^{\frac{m}{m-1}}$$

with $m = \frac{4k}{n(q-1)}$ to the right side of (6) we verify that

$$(7) \quad \int_\Omega |v(x)|^{2+\frac{q-1}{k}} dx \leq \frac{c}{m} \bar{\varepsilon}^m \left[\delta_{1,2} \int_\Omega v^2(x)\ dx + \right.$$

$$\left. + \int_\Omega \sum_{i=1}^n (v_{x_i})^2\ dx \right] + c\ \frac{m-1}{m} \left[\bar{\varepsilon}^{-1} \int_\Omega v^2(x)\ dx \right]^{\frac{m}{m-1}},$$

where $\frac{m}{m-1} = \frac{4k}{4k-n(q-1)}$ and $\delta_{1,2}$ is equal to 0 for the Dirichlet problem and to 1 for the Neumann problem. Next formula (7) with $\bar{\varepsilon} = \bar{\varepsilon}_0$ such that

$$\frac{c}{m} \bar{\varepsilon}_0^m = \frac{2k-1}{k^2}$$

will be used to estimate the last component in (5) and to get:

$$\frac{d}{dt} \int_\Omega u^{2k}(t,x)\ dx \leq \frac{4k-2}{k}\ \delta_{1,2} \int_\Omega u^{2k}(t,x)\ dx +$$

$$+ r^{-1} \lambda c \left[\frac{kcn(q-1)}{4(2k-1)}\right]^{r-1} 2k \left(\int_\Omega u^{2k}(t,x)\ dx\right)^r,$$

where $r := \frac{4k}{4k-n(q-1)}$. This simple differential inequality for functions $y_k(t) := \int_\Omega u^{2k}(t,x)dx;\ k \in N$, has the form

$$y_k'(t) \leq \frac{4k-2}{k}\ \delta_{1,2}\ y_k(t) + \text{const.}(k)\ 2k\ (y_k(t))^r,$$

and can be solved explicitly, to get:

$$(8') \qquad y_k^{1-r}(t) \geq y_k^{1-r}(0) + \text{const.}(k)\ 2k(1-r)t$$

for the Dirichlet problem ($\delta_{1,2} = 0$) and

(8″)
$$y_k^{1-r}(t) \geq$$
$$\geq \left[y_k^{1-r}(0) + \text{const.}(k)\ 2k\ \frac{k}{4k-2}\left(1 - \exp\left(\frac{(r-1)(4k-2)t}{k}\right)\right)\right]$$
$$\exp\left(\frac{(1-r)(4k-2)t}{k}\right),$$

for the Neumann problem ($\delta_{1,2} = 1$). Carefully controlling the constants and remembering that $y_k(t) = \|u(t,\cdot)\|^{2k}_{L^{2k}(\Omega)}$, we can pass in (8′), (8″) with k to infinity and get

$$(9) \qquad \|u(t,\cdot)\|^{\frac{n(q-1)}{2}}_{L^\infty(\Omega)} \leq \left[\|u_0\|^{\frac{n(q-1)}{2}}_{L^\infty(\Omega)} - \frac{\lambda cn}{2}(q-1)t\right]^{-1}$$

for both Dirichlet and Neumann problems. The a priori estimate (9) is valid until the first time $t = t_1$ for which the bracket is equal to zero. Clearly this t_1 estimates the existence time T of the solution u from below. We must point out that (9) is only an estimate of the $L^\infty(\Omega)$ norm of u from above. If we are able to give an a priori estimate of this norm valid for all $t \in [0,t_2)$, with $t_2 \in (t_1,T]$, then all further considerations in the proof of existence work for t_2 instead of t_1. In particular for solutions considered in Lemma 1 we have $t_2 = T = +\infty$, and this solutions exist for all $t \geq 0$ (are global solutions).

The proof of existence of the uniformly Hölder continuous solution of (1)-(3) is now standard (compare [3], [4]) and will be finished in two steps. We sketch it briefly. The first step contains an a priori estimate of the $L^{2n+2}(\Omega)$ norm of the derivative u_t. We avoid here formal consideration of the behaviour of difference quotient $\frac{u(t+h,x)-u(t,x)}{h}$ (compare [3]), showing instead more clearly the idea of the estimate. Differentiating (1) with respect to t, multiplying the result by u^{2n+1} and integrating, we have

$$(10)\qquad \frac{1}{2n+2}\frac{d}{dt}\int_\Omega u_t^{2n+2}\,dx =$$
$$= -\frac{2n+1}{(n+1)^2}\int_\Omega \sum_{i=1}^{n}\left[(u_t)^{n+1}\right]^2_{x_i}\,dx +$$
$$+ \lambda q\int_\Omega |u|^{q-1}\,u_t^{2n+2}\,dx.$$

Fixing an arbitrary $\varepsilon\ (0<\varepsilon<t_1)$ we will estimate the last component in (10) for $t\in[0,t_1-\varepsilon]$, with the use of (9), as follows

$$\lambda q\int_\Omega |u|^{q-1}\,u_t^{2n+2}\,dx \leqslant \lambda q\,M^{q-1}(\varepsilon)\int_\Omega u_t^{2n+2}\,dx$$

($M(\varepsilon)$ dominates $\|u(t,\cdot)\|_{L^\infty(\Omega)}$ for $t\in[0,t_1-\varepsilon]$), hence, neglecting the first right side component in (10), we find that

$$(11)\quad \int_\Omega u_t^{2n+2}(t,x)dx \leqslant \int_\Omega u_t^{2n+2}(0,x)dx\,\exp\left[(2n+2)\,\lambda q\,M^{q-1}(\varepsilon)t\right],$$

for every $t\in[0,t_1-\varepsilon]$. For smooth solutions considered here $u_t(0,x)$ can be found from (1) and estimated as below

$$\|u_t(0,\cdot)\|_{L^{2n+2}(\Omega)} \leqslant \|u_0\|_{W^{2,2n+2}(\Omega)} + \lambda|\Omega|^{\frac{1}{2n+2}}\,\|u_0\|^q_{L^\infty(\Omega)},$$

where $|\Omega|$ denotes the measure of Ω.

The estimates (9) and (11) generate several a priori estimates for Hölder norms of solutions of (1)-(3). Let $u\in C^{1+\frac{\alpha}{4},2+\frac{\alpha}{2}}([0,t_1-\varepsilon]\times\bar\Omega)$ be such a solution, and consider (1) with fixed $t>0$ (here a parameter) as an elliptic equation:

$$(12)\qquad \Delta u(t,\cdot) = u_t(t,\cdot) - \lambda|u(t,\cdot)|^{q-1}\,u(t,\cdot).$$

The right side of (12) is estimated for $t\in[0,t_1-\varepsilon]$ in the $L^{2n+2}(\Omega)$ norm in formulae (9) and (11). As a consequence of the Calderon-Zygmund estimates [10] for solutions of elliptic equations

$$\|u(t,\cdot)\|_{W^{2,2n+2}(\Omega)} \leqslant c'\|u_t(t,\cdot)-\lambda|u(t,\cdot)|^{q-1}u(t,\cdot)\|_{L^{2n+2}(\Omega)} \leqslant$$

$$\leqslant \text{const.} \quad \text{when} \quad t\in[0,t_1-\varepsilon].$$

In other words, u is a priori bounded in the space $L^\infty([0,t_1-\varepsilon];W^{2,2n+2}(\Omega))$, also as a consequence of (11) u_t is bounded a priori in $L^\infty([0,t_1-\varepsilon];L^{2n+2}(\Omega))$. These last observations also ensure that the gradient $(u_t,\nabla_x u)$ is bounded in $L^\infty([0,t_1-\varepsilon];L^{2n+2}(\Omega))\subset L^{2n+2}([0,t_1-\varepsilon]\times\Omega)$. From the Sobolev Imbedding Theorem [10], [8];

$$W^{k,p}(B) \hookrightarrow C^{j+\mu}(\bar{B}) \quad \text{for} \quad 0<\mu = k-\frac{m}{p}-j<1,\ m=\dim B,$$

applied to u (here $m = n+1$), this together with (9) gives the fundamental estimate in Holder norms

$$\|u\|_{C^{\frac{1}{2},\frac{1}{2}}([0,t_1-\varepsilon]\times\bar{\Omega})} \leqslant \text{const.} \tag{13}$$

Now the nonlinear term $\lambda|u|^{q-1}$ in (1) will be considered as the "uniformly Hölder continuous coefficient" (the function $|u|^{q-1}$ is the composition of the $C^{\frac{1}{2},\frac{1}{2}}$ Hölder continuous function u, Lipschitz continuous absolute value and locally Lipschitz continuous $(q-1)$ power) belonging to $C^{\frac{1}{2},\frac{1}{2}}([0,t_1-\varepsilon]\times\bar{\Omega})$. The standard use of the Leray-Schauder Principle (c.f. [9], [4]) finishes the proof of existence of the solution.

R e m a r k 1. It is a simple consequence of the Maximum Principle (compare [11]), that whenever $u_0(x)\geqslant 0$ for $x\in\bar{\Omega}$, then the classical solution u of (1)-(3) is nonnegative as long as it exists. Such solutions simultaneously solve the problem (4), (2), (3).

3. Behaviour of solutions

We have the following:

Theorem 2. For sufficiently small in $L^\infty(\Omega)$ initial function u_0 the corresponding solution u of (1), (2), (3′) exists for all $t \geq 0$ and tends uniformly to zero when t tends to infinity.

Proof. Let μ_1 be the first positive eigenvalue of $-\Delta$ under homogeneous Dirichlet conditions. We change the nonlinear term in (1) outside the neighbourhood of zero, putting:

$$
(14)\qquad f(z) = \begin{cases} \lambda|z|^{q-1}z & \text{for } z \in X := \left\{z;\ |z| < \left(\frac{\mu_1-\varepsilon_1}{\lambda}\right)^{\frac{1}{q-1}}\right\}, \\ \lambda\left(\frac{\mu_1-\varepsilon_1}{\lambda}\right)^{\frac{1}{q-1}} \operatorname{sign}(z) & \text{for } z \notin X, \end{cases}
$$

($\varepsilon_1 \in (0,\mu_1)$ is fixed). Note also that a nonlinear function f of this kind is uniformly Lipschitz continuous and satisfies

$$f(u)\cdot u \leq (\mu_1-\varepsilon_1)\ u^2.$$

Consider the Dirichlet problem (2), (3′) for the equation

$$(15)\qquad w_t = \Delta w + f(w)$$

and note that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2(t,x)dx = -\int_\Omega \sum_{i=1}^{n} (w_{x_i})^2 dx + (\mu_1 - \varepsilon_1)\int_\Omega w^2 dx \leq$$

$$\leq -\varepsilon_1 \int_\Omega w^2(t,x)dx,$$

where we have used the Poincaré inequality

$$\forall_{v\in H_0^1(\Omega)}\quad \mu_1\|v\|^2_{L^2(\Omega)} \leq \|\nabla_x v\|^2_{L^2(\Omega)}.$$

Hence

$$\int_\Omega w^2(t,x)dx \leqslant \int_\Omega w^2(0,x)dx \exp(-2\varepsilon_1 t). \tag{16}$$

For the problem (15), (2), (3′) all the assumptions stated in [4] are satisfied, and as a consequence of Theorem 3.1,

$$\|w(t,\cdot)\|_{L^\infty(\Omega)} \leqslant \tag{17}$$

$$\leqslant K \max\left\{\|w(0,\cdot)\|_{L^\infty(\Omega)};\ \sup_{t\geqslant 0}\|w(t,\cdot)\|_{L^2(\Omega)}\right\},$$

with $K = K(\Omega,\lambda,\mu_1,\varepsilon_1)$. Moreover, the solution w exists for all $t \geqslant 0$ and $\|w(t,\cdot)\|_{L^\infty(\Omega)} \to 0$ when t tends to infinity.

Now, as a consequence of (14), (16) and (17), when

$$\|w(0,\cdot)\|_{L^\infty(\Omega)} \leqslant \frac{1}{K\max\left\{1;|\Omega|^{\frac{1}{2}}\right\}}\left(\frac{\mu_1-\varepsilon_1}{\lambda}\right)^{\frac{1}{q-1}}, \tag{18}$$

then $w(t,x) \in X$ for all $(t,x) \in [0,\infty) \times \bar{\Omega}$. For such values of w the equation (15) coincides with (1). This means that w solves the original problem (1), (2), (3′) and finishes the proof.

We may now review several simple methods of verification that the blow up really takes place (compare [6]). The first two methods concern the case of nonnegative solutions, i.e. solutions of the problem (4), (2), (3).

Consider the Dirichlet type condition. Let v_1 be the first nonnegative (compare [4]) eigenfunction corresponding to the eigenvalue μ_1 of the elliptic problem

$$\begin{aligned} &\Delta v + \mu_1 v = 0 \\ &v = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{19}$$

Multiplying v_1 by $(\sup_\Omega v_1(x))^{-1}$ we may additionally assume that $0 \leqslant v_1(x) \leqslant 1$. We have

T h e o r e m 3. Under the conditions: $u_0(x) \geq 0$ and

$$\int_\Omega u_0(x)\ v_1(x)dx > \left(\frac{\mu_1 c_{1q}^q}{\lambda}\right)^{q-1}, \tag{20}$$

the solution u of (1), (2), (3′) corresponding to u_0 blows up. Here c_{1q} is the constant for which $\|\varphi\|_{L^1(\Omega)} \leq c_{1q}\|\varphi\|_{L^q(\Omega)}$ for all φ in $L^q(\Omega)$.

P r o o f . Remembering that $u(t,x) \geq 0$, then multiplying (1) by v_1 and integrating over Ω we get

$$\frac{d}{dt}\int_\Omega u(t,x)\ v_1(x)dx = -\mu_1 \int_\Omega u(t,x)\ v_1(x)dx +$$

$$+ \lambda\int_\Omega u^q(t,x)\ v_1(x)dx.$$

Now, since $0 \leq v_1(x) \leq 1$, we have $v_1(x) \geq v_1^q(x)$, and

$$\frac{d}{dt} T_1(t) \leq -\mu_1\ T_1(t) + \lambda\, c_{1q}^{-q}\left[T_1(t)\right]^q, \tag{21}$$

where $T_1(t) := \int_\Omega u(t,x)\ v_1(x)dx$ is the first Fourier coefficient of u. Every solution of the differential inequality (21) with the initial value satisfying (20) blows up in finite fime.

Consider now the nonnegative solution of the Neumann problem.

T h e o r e m 4. Every nonnegative ($\not\equiv 0$) solutions of the problem (4), (2), (3″) blows up in finite time.

P r o o f . Integrating (4) over Ω we get:

$$\frac{d}{dt}\int_\Omega u(t,x)dx = -\int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds + \lambda\int_\Omega u^q(t,x)dx \geq$$

$$\geq c_{1q}^{-q}\ \lambda\left(\int_\Omega u(t,x)dx\right)^q.$$

Solving this differential inequality we obtain

$$(22)\qquad \Big(\int_\Omega u(t,x)dx\Big)^{q-1} \geqslant \Big[\Big(\int_\Omega u_0(x)dx\Big)^{q-1} - (q-1)\lambda c_{1q}t\Big]^{-1},$$

and this estimate finishes the proof.

The same result is valid for nonpositive and non-vanishing solutions of (1), (2), (3″).

The last method holds good for solutions of arbitrary sign and both homogeneous Dirichlet or Neumann conditions. The idea of the proof is derived from [7] p.419.

T h e o r e m 5. Every solution of (1)-(3) where the $L^{q+1}(\Omega)$ norm of u_0 is large relative to the $L^2(\Omega)$ norm of ∇u_0 (condition (27)), blows up in finite time.

P r o o f . Multiplying (1) by u and by $2u_t$ and integrating, we get:

$$(23)\qquad \frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx = -\int_\Omega \sum_{i=1}^n (u_{x_i})^2dx + \lambda\int_\Omega |u|^{q+1}dx,$$

and

$$(24)\qquad 2\int_\Omega u_t^2dx = -\frac{d}{dt}\Big[\int_\Omega \sum_{i=1}^n (u_{x_i})^2dx - \frac{2\lambda}{q+1}\int_\Omega |u|^{q+1}dx\Big] =:$$

$$=: -\frac{d}{dt}E(t).$$

The function E in the square bracket in (24) is nonincreasing, hence

$$(25)\qquad \int_\Omega \sum_{i=1}^n (u_{x_i})^2dx \leqslant E(0) + \frac{2\lambda}{q+1}\int_\Omega |u|^{q+1}dx.$$

Combining (23) with (25) we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx \geqslant -E(0) + s\int_\Omega |u|^{q+1}dx$$

with $s = \lambda\left(1 - \frac{2}{q+1}\right)$ always positive $(q > 1)$. Denoting $z(t) :=$ $:= \int_\Omega u^2(t,x)dx$, since $\|v\|_{L^2(\Omega)} \leq c_{2(q+1)}\|v\|_{L^{q+1}(\Omega)}$, we finally verify that

$$z'(t) \geq -2\,E(0) + \frac{2s}{c_{2(q+1)}^{q+1}}\,(z(t))^{\frac{q+1}{2}}. \tag{26}$$

It is easy to see that every solution of (26) with $z(0) =$ $= \int_\Omega u_0^2(x)dx$ satisfying

$$z(0)^{\frac{q+1}{2}} > \frac{E(0)\ c_{2(q+1)}^{q+1}}{s} \equiv \tag{27}$$

$$c_{2(q+1)}^{q+1}\left[\lambda\left(1 - \frac{2}{q+1}\right)\right]^{-1}\left[\int_\Omega \sum_{i=1}^{n} (u_0)_{x_i}^2 dx - \frac{2\lambda}{q+1}\int_\Omega |u_0|^{q+1}dx\right]$$

blows up in finite time.

R e m a r k 2. The left side of the inequality (27) will be estimated from above by $c_{2(q+1)}^{q+1}\int_\Omega |u_0|^{q+1}dx$ and the final condition takes the form

$$\lambda\int_\Omega |u_0|^{q+1}dx > \int_\Omega \sum_{i=1}^{n} (u_0)_{x_i}^2 dx. \tag{28}$$

The above estimate could not hold, in the case of the Dirichlet condition (3′), for small initial functions; for any u_0 satisfying $|u_0(x)| \leq \delta < 1$ with $\delta^{q-1} < \mu_1\lambda^{-1}$, as a consequence of the Poincaré inequality, holds

$$\int_\Omega |u_0|^{q+1}dx \leq \delta^{q-1}\int_\Omega u_0^2 dx \leq \frac{\delta^{q-1}}{\mu_1}\int_\Omega \sum_{i=1}^{n} (u_0)_{x_i}^2 dx,$$

and we have an estimate exactly opposite to (27).

R e m a r k 3. The final picture of the behaviour of solutions of (1)-(3) is the following. For the Neumann problem, every non-vanishing solution of constant sign blows up (Theorem 4). The same is true for solutions satisfying the condition (27). For the Dirichlet problem, with small initial data the solution is global and tends to zero when t goes to infinity (Theorem 2), while for a large initial function it blows up (Theorem 5 and 3).

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INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY,
40-007 KATOWICE, POLAND

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