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EQUALITY IN MINKOWSKI INEQUALITY AND A CHARACTERIZATION OF L^p-NORM

Introduction

For a measure space (Ω, Σ, μ) denote by $\boldsymbol{S} = \boldsymbol{S}(\Omega, \Sigma, \mu)$ the linear space of all μ-*integrable* simple functions $x : \Omega \mapsto \mathbf{R}$, and by $\boldsymbol{S}_+ = \boldsymbol{S}_+(\Omega, \Sigma, \mu)$ the set of all nonnegative $x \in \boldsymbol{S}(\Omega, \Sigma, \mu)$. It is easy to see that for an arbitrary bijection $\phi : (0, \infty) \mapsto (0, \infty)$ the functional $\boldsymbol{p}_\phi : \boldsymbol{S} \mapsto [0, \infty)$ given by

$$\boldsymbol{p}_\phi(x) := \begin{cases} \phi^{-1}\Big(\int\limits_{\Omega(x)} \phi \circ |x| \, d\mu \Big) & \text{if } \mu(\Omega(x)) > 0 \\ 0 & \text{if } \mu(\Omega(x)) = 0 \end{cases}, \quad x \in \boldsymbol{S}(\Omega, \Sigma, \mu),$$

where $\Omega(x) := \{\omega \in \Omega : x(\omega) \neq 0\}$, is correctly defined i.e. for every $x \in \boldsymbol{S}(\Omega, \Sigma, \mu)$ the function $\phi \circ |x|$ is Σ–measurable and the integral $\int_{\Omega(x)} \phi \circ |x| d\mu$ is finite (cf. [4], Remark 5).

Note that for $\phi(t) := \phi(1)t^p$, $t > 0$, where $p \in \mathbf{R} \setminus \{0\}$ is arbitrary and fixed, we have

$$\boldsymbol{p}_\phi(x) = \Big(\int\limits_{\Omega(x)} |x|^p \, d\mu \Big)^{\frac{1}{p}}, \quad x \in \boldsymbol{S}(\Omega, \Sigma, \mu), \ \mu(\Omega(x)) > 0.$$

For $p \geq 1$ the functional $\boldsymbol{p}_\phi$ becomes the $\mathbf{L}^p$-norm. So, for $p \geq 1$ we have the Minkowski inequality

$$\boldsymbol{p}_\phi(x + y) \leq \boldsymbol{p}_\phi(x) + \boldsymbol{p}_\phi(y), \quad x, y \in \boldsymbol{S}(\Omega, \Sigma, \mu),$$

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and for $p < 1$, $p \neq 0$, the reversed "companion inequality"

$$\boldsymbol{p}_\phi(x+y) \geq \boldsymbol{p}_\phi(x) + \boldsymbol{p}_\phi(y), \quad x, y \in \boldsymbol{S}_+(\Omega, \Sigma, \mu).$$

It is well known that, in each of these inequalities, the equality occurs if, and only if, the functions x and y are positively proportional, i.e. there is a $t > 0$ such that $y = tx$ ($\mu - a.e.$). It turns out that this fact allows to characterize the $\mathbf{L}^p$-norm.

We shall prove that if, for a class of measure spaces, a bijection $\phi : (0,\infty) \mapsto (0,\infty)$ satisfies the condition,

$$(\star) \qquad \boldsymbol{p}_\phi(x+tx) = \boldsymbol{p}_\phi(x) + \boldsymbol{p}_\phi(tx), \quad x \in \boldsymbol{S}(\Omega, \Sigma, \mu),\ t > 0,$$

then the function $\psi(t) := \phi(t)/\phi(1)$, $t > 0$, is multiplicative. Hence, under some weak regularity conditions, ψ must be a power function. Since the condition $(\star)$ is an obvious consequence of the positive homogeneity of the functional $\boldsymbol{p}_\phi$,

$$\boldsymbol{p}_\phi(tx) = t\boldsymbol{p}_\phi(x), \quad x \in \boldsymbol{S}(\Omega, \Sigma, \mu),\ t > 0,$$

one of the results of this paper (Theorem 2) implies some characterizations of the $\mathbf{L}^p$-norms given by Zaanen [6], Wnuk [5], and J. Matkowski [2].

1. Auxiliary results

Remark 1. Suppose that $\mu(\Omega) > 0$ and take an arbitrary $x \in \boldsymbol{S}_+$ such that $\mu(\Omega(x)) > 0$.

Then there exist pairwise disjoint sets $A_1, \ldots, A_n \in \Sigma$, of finite and positive measure, and $x_1, \ldots, x_n > 0$, such that

$$x = \sum_{i=1}^{n} x_i \chi_{A_i},$$

(here χ_A stands for the characteristic function of A).

By the definition of $\boldsymbol{p}_\phi$ we get

$$\boldsymbol{p}_\phi(x) = \phi^{-1}\Big(\sum_{i=1}^{n} \phi(x_i)\mu(A_i)\Big).$$

In the sequel the following lemma plays an essential role (cf. the proof of Theorem 1 in [3]):

LEMMA 1. *Let $\phi : (0,\infty) \mapsto (0,\infty)$ be an arbitrary bijection. Then the function $\phi^{-1} \circ (a\phi)$ is additive for every fixed $a > 0$ if, and only if, the function $\psi : (0,\infty) \mapsto (0,\infty)$, defined by*

$$\psi(t) := \frac{\phi(t)}{\phi(1)}, \qquad t > 0,$$

is multiplicative.

Proof. Suppose that for every fixed $a > 0$ the function $\phi^{-1} \circ (a\phi)$ is additive. Since $\phi^{-1} \circ (a\phi)$ is positive, it must be linear. Thus there exists a function $m : (0,\infty) \mapsto (0,\infty)$ such that

$$\phi^{-1}[a\phi(u)] = m(a)u, \qquad u > 0;\ a > 0.$$

Replacing a by b, we have

$$\phi^{-1}[b\phi(u)] = m(b)u, \qquad u > 0;\ b > 0.$$

Composing separately the functions on the left and on the right–hand sides of the above equations gives

$$\phi^{-1}[ab\phi(u)] = m(a)m(b)u, \qquad u > 0;\ a, b > 0.$$

On the other hand we also have

$$\phi^{-1}[ab\phi(u)] = m(ab)u, \qquad u > 0;\ a, b > 0,$$

and, consequently,

$$m(ab) = m(a)m(b), \qquad a, b > 0,$$

which means that $m : (0,\infty) \mapsto (0,\infty)$ is multiplicative. Since $\phi^{-1}[a\phi(u)] = m(a)u$, $u > 0$, $a > 0$, we have

$$m(a) = \phi^{-1}[a\phi(1)], \qquad a > 0.$$

It follows that m is bijective, and consequently, the inverse function m^{-1} : $(0,\infty) \mapsto (0,\infty)$,

$$m^{-1}(a) = \frac{\phi(a)}{\phi(1)}, \qquad a > 0,$$

is multiplicative. Thus the function ψ is multiplicative.

Suppose that ψ is multiplicative. Then the inverse function ψ^{-1},

$$\psi^{-1}(u) = \phi^{-1}[\phi(1)u], \qquad u > 0,$$

is multiplicative. Consequently, for a fixed and arbitrary $a > 0$, and for all $s, t > 0$, we have

$$\phi^{-1}[a\phi(s+t)] = \phi^{-1}\left[\phi(1)a\frac{\phi(s+t)}{\phi(1)}\right] = \psi^{-1}[a\psi(s+t)]$$

$$= \psi^{-1}(a)\psi^{-1}[\psi(s+t)] = \psi^{-1}(a)s + \psi^{-1}(a)t$$
$$= \psi^{-1}(a)\psi^{-1}[\psi(s)] + \psi^{-1}(a)\psi^{-1}[\psi(t)]$$
$$= \psi^{-1}[a\psi(s)] + \psi^{-1}[a\phi(t)] = \phi^{-1}[a\phi(s)] + \phi^{-1}[a\phi(t)],$$

which completes the proof.

2. Some results for the case when the union of of ranges of the admitted measures contains $(0,\infty)$

We begin with the following

THEOREM 1. *Suppose that $\phi : (0,\infty) \mapsto (0,\infty)$ is bijective. Then the following conditions are equivalent:*

1^0. *the function $\psi : (0,\infty) \mapsto (0,\infty)$,*

$$\psi(t) := \frac{\phi(t)}{\phi(1)}, \qquad t > 0,$$

is multiplicative;

2^0. *for every measure space (Ω, Σ, μ), the functional $\boldsymbol{p}_\phi$ is positively homogeneous, i.e.*

$$\boldsymbol{p}_\phi(tx) = t\boldsymbol{p}_\phi(x), \qquad x \in \boldsymbol{S}(\Omega, \Sigma, \mu),\ t > 0;$$

3^0. *for every measure space (Ω, Σ, μ),*

$$\boldsymbol{p}_\phi(x + tx) = \boldsymbol{p}_\phi(x) + \boldsymbol{p}_\phi(tx), \qquad x \in \boldsymbol{S}(\Omega, \Sigma, \mu),\ t > 0;$$

4^0. *there is a family of measure spaces $((\Omega_i, \Sigma_i, \mu_i))_{i\in I}$ such that*

$$(0,\infty) \subset \bigcup_{i\in I} \mu_i(\Sigma_i),$$

and, for every $i \in I$,

$$\boldsymbol{p}_\phi(x + tx) = \boldsymbol{p}_\phi(x) + \boldsymbol{p}_\phi(tx), \qquad x \in \mathbf{S}_+(\Omega_i, \Sigma_i, \mu_i); \qquad t > 0.$$

Proof. First note that ψ is bijective,

$$\phi^{-1}(t) = \psi^{-1}\left(\frac{t}{\phi(1)}\right), \qquad t > 0,$$

and, for all $x \in \mathbf{S}$ we have

$$\boldsymbol{p}_\phi(x) := \phi^{-1}\Big(\int\limits_{\Omega(x)} \phi\circ|x|\,d\mu\Big) = \psi^{-1}\Big(\frac{1}{\phi(1)}\int\limits_{\Omega(x)} \phi(1)\psi\circ|x|\,d\mu\Big) = \mathbf{p}_\psi(x).$$

Suppose now that condition 1^0 is fulfilled, i.e. that ψ is multiplicative. Then, clearly, ψ^{-1} is multiplicative. If (Ω,Σ,μ) is an arbitrary measure space, we have for all $t>0$ and $x\in \boldsymbol{S}(\Omega,\Sigma,\mu)$,

$$\begin{aligned}\boldsymbol{p}_\phi(tx) = \mathbf{p}_\psi(tx) &= \psi^{-1}\Big(\int\limits_{\Omega(x)} \psi\circ(|tx|)\,d\mu\Big)\\ &= \psi^{-1}\Big(\int\limits_{\Omega(x)} \psi(t)\psi\circ(|x|)\,d\mu\Big)\\ &= \psi^{-1}\Big(\psi(t)\int\limits_{\Omega(x)} \psi\circ(|x|)\,d\mu\Big)\\ &= \psi^{-1}(\psi(t))\psi^{-1}\Big(\int\limits_{\Omega(x)} \psi\circ(|x|)\,d\mu\Big)\\ &= t\mathbf{p}_\psi(x) = t\boldsymbol{p}_\phi(x),\end{aligned}$$

which shows that condition 1^0 implies 2^0.

Suppose condition 2^0. Then, for all $t>0$ and $x\in \boldsymbol{S}_+(\Omega,\Sigma,\mu)$,

$$\begin{aligned}\boldsymbol{p}_\phi(x+tx) = \boldsymbol{p}_\phi((1+t)x) &= (1+t)\boldsymbol{p}_\phi(x)\\ &= \boldsymbol{p}_\phi(x) + t\boldsymbol{p}_\phi(x) = \boldsymbol{p}_\phi(x) + \boldsymbol{p}_\phi(tx),\end{aligned}$$

i.e. the condition 3 holds true.

The implication "$3^0 \Rightarrow 4^0$" is obvious.

Suppose that condition 4^0 holds true. Take an arbitrary $a>0$. Then there exist $i\in I$, and $A\in\Sigma_i$ such that $a=\mu_i(A)$. Let $s>0$ be arbitrary. Since $x := s\chi_A \in \boldsymbol{S}(\Omega,\Sigma,\mu)_+(\Omega_i,\Sigma_i,\mu_i)$, we have

$$\boldsymbol{p}_\phi[x+(ts^{-1})x] = \boldsymbol{p}_\phi(x) + \boldsymbol{p}_\phi(ts^{-1}x), \qquad s,t>0,$$

which, by Remark 1, can be written in the form

$$\phi^{-1}[a\phi(s+t)] = \phi^{-1}[a\phi(s)] + \phi^{-1}[a\phi(t)], \qquad s,t>0.$$

Thus, for every $a>0$, the function $\phi^{-1}\circ(a\phi)$ is additive. By Lemma 1, the function ψ is multiplicative. This completes the proof.

Remark 2. Let $(\Omega_a,\Sigma_a,\mu_a)$ be a measure space such that

$$\Omega_a := \{1\}, \qquad \Sigma_a := \{\emptyset,\{1\}\}, \qquad \mu_a(\{1\}) = a.$$

Then the family $((\Omega_a, \Sigma_a, \mu_a))_{a\in(0,\infty)}$ satisfies the assumption of condition 4^0.

Remark 3. Note that each singleton family of measure spaces $((\Omega, \Sigma, \mu))$ such that $(0,\infty) \subset \mu(\Sigma)$ fulfils all the assumptions of condition 4^0.

COROLLARY 1. *Let* $\phi : (0,\infty) \mapsto (0,\infty)$ *be bijective. If* ϕ *is measurable, or* $\log \circ \phi$ *is bounded above (below) in neighbourhood of a point, then the condition*

1^0. $\phi(t) = \phi(1)t^p$, $t > 0$, *for some* $p \in \mathbf{R}$, $p \neq 0$;

is equivalent to each of the conditions 2^0–4^0 *of Theorem* 1.

3. A result for measure spaces with at least two disjoint sets of finite positive measure

The main result of this section reads as follows:

THEOREM 2. *Let* (Ω, Σ, μ) *be a measure space with* $A, B \in \Sigma$ *such that* $A \cap B = \emptyset$, *and* $\mu(A), \mu(B)$ *are positive and finite. Suppose that* $\phi : (0,\infty) \mapsto (0,\infty)$ *is bijective, and* ϕ *or* ϕ^{-1} *is continuous at least at one point. If*

$$\mathbf{p}_\phi(x + tx) = \mathbf{p}_\phi(x) + \mathbf{p}_\phi(tx), \qquad x \in \mathbf{S}_+(\Omega, \Sigma, \mu),\ t > 0; \tag{1}$$

then $\phi(t) = \phi(1)t^p$, $t > 0$, *for some* $p \in \mathbf{R}$, $p \neq 0$.

Proof. Put $a := \mu(A)$, $b := \mu(B)$. Taking $x = s\chi_A$ and replacing t by $s^{-1}t$ in (1) gives

$$\phi^{-1}[a\phi(s+t)] = \phi^{-1}[a\phi(s)] + \phi^{-1}[a\phi(t)], \qquad s,t > 0.$$

Thus there is a constant $\alpha > 0$ such that $\phi^{-1}[a\phi(u)] = \alpha u$, $u > 0$. Hence

$$a\phi(u) = \phi(\alpha u), \qquad u > 0. \tag{2}$$

In the same way one can show that there is a $\beta > 0$ such that

$$b\phi(u) = \phi(\beta u), \qquad u > 0. \tag{3}$$

Substituting $x = u\chi_A + v\chi_B$, with positive u, v in (1) gives

$$\phi^{-1}[a\phi(u+tu) + b\phi(v+tv)] = \phi^{-1}[a\phi(u) + b\phi(v)] + \phi^{-1}[a\phi(tu) + b\phi(tv)].$$

Making use of (2) and (3) we can write this equation in the form

$$\phi^{-1}[\phi(\alpha u + t\alpha u) + \phi(\beta v + t\beta v)] = \phi^{-1}[\phi(\alpha u) + \phi(\beta v)] + \phi^{-1}[\phi(t\alpha u) + \phi(t\beta v)].$$

Replacing αu and βv, respectively, by u and v we obtain

$$(4)\quad \phi^{-1}[\phi(u+tu)+\phi(v+tv)] = \phi^{-1}[\phi(u)+\phi(v)] + \phi^{-1}[\phi(tu)+\phi(tv)],$$

for all $t, u, v > 0$.

We shall prove that for every $k \in \mathbf{N}$,

$$(5)\qquad \phi^{-1}[\phi(ku)+\phi(kv)] = k\phi^{-1}[\phi(u)+\phi(v)], \qquad u, v > 0.$$

This relation is obvious for $k = 1$. Suppose it is true for a $k \in \mathbf{N}$. Then, taking $t = k$ in (4) gives

$$\begin{aligned}\phi^{-1}[\phi((k+1)u)+\phi((k+1)v)] &= \phi^{-1}[\phi(u+ku)+\phi(v+kv)] \\ &= \phi^{-1}[\phi(u)+\phi(v)] + \phi^{-1}[\phi(ku)+\phi(kv)] \\ &= (k+1)\phi^{-1}[\phi(u)+\phi(v)],\end{aligned}$$

and, by induction, (5) holds true for all $k \in \mathbf{N}$ and $u, v > 0$.

Taking the value ϕ of both sides, and then replacing u by $\phi^{-1}(u)$, and v by $\phi^{-1}(v)$, one gets

$$\phi[k\phi^{-1}(u)] + \phi[k\phi^{-1}(v)] = \phi[k\phi^{-1}(u+v)], \qquad u, v > 0;\ k \in \mathbf{N},$$

which shows that for every $k \in \mathbf{N}$ the function $\phi \circ (k\phi^{-1})$ is additive, and consequently, linear. In particular, for $k = 2$, and $k = 3$ there are $\gamma > 0$, $\gamma \neq 1$, and $\delta > 0$, $\delta \neq 1$, such that

$$\phi[2\phi^{-1}(u)] = \gamma u, \qquad \phi[3\phi^{-1}(u)] = \delta u, \qquad u > 0.$$

It follows that ϕ satisfies the simultaneous system of the functional equations

$$\phi(2u) = \gamma\phi(u), \qquad \phi(3u) = \delta\phi(u), \qquad u > 0.$$

Since $\log 3/\log 2$ is irrational, the continuity of ϕ at least at one point implies that $\phi(u) = \phi(1)u^p$, $u > 0$, for some $p \in \mathbf{R}$ (cf. [1]).

Similarly, the function ϕ^{-1} satisfies the simultaneous system of functional equations

$$\phi^{-1}(\gamma u) = 2\phi^{-1}(u), \qquad \phi^{-1}(\delta u) = 3\phi^{-1}(u), \qquad u > 0.$$

Again, since $\log 3/\log 2$ is irrational, the continuity of ϕ^{-1} at least at one point implies that, up to a factor, ϕ is a power function (cf. [1]). This completes the proof.

Remark 4. Replacing u by $k^{-1}u$, and v by $k^{-1}v$ in (5) gives

$$\phi^{-1}[\phi(k^{-1}u)+\phi(k^{-1}v)]=k^{-1}\phi^{-1}[\phi(u)+\phi(v)], \quad u,v>0. \tag{6}$$

Now from (5) and (6) we obtain

$$\phi^{-1}[\phi(k^{-1}nu)+\phi(k^{-1}nv)]=k^{-1}n\phi^{-1}[\phi(u)+\phi(v)], \quad u,v>0;\ k,n\in\mathbf{N},$$

which means that

$$\phi^{-1}[\phi(ru)+\phi(rv)]=r\phi^{-1}[\phi(u)+\phi(v)], \quad u,v>0; \quad r\in\mathbf{Q}_+,$$

where $\mathbf{Q}_+$ denotes the set of all positive rational numbers. Thus the function of two variables $F(u,v):=\phi^{-1}[\phi(u)+\phi(v)]$ is rationally positively homogeneous without any regularity assumptions.

4. A result for a measure space with at least ons set of finite positive measure

In this section we assume that the function ϕ satisfies some asymptotic conditions.

THEOREM 3. *Let (Ω,Σ,μ) be a measure space with a set $A\in\Sigma$ such that $0<\mu(A)<\infty$, and $\mu(A)\neq 1$. Suppose that $\phi:(0,\infty)\mapsto(0,\infty)$ is bijective and that there exists a $p\in\mathbf{R}$, $p\neq 0$, such that one of the limits*

$$\lim_{u\to 0+}\frac{\phi(u)}{u^p}, \qquad \lim_{u\to\infty}\frac{\phi(u)}{u^p},$$

exists and is a positive real number. If

$$\boldsymbol{p}_\phi(x+tx)=\boldsymbol{p}_\phi(x)+\boldsymbol{p}_\phi(tx), \quad x\in\boldsymbol{S}_+(\Omega,\Sigma,\mu),\ t>0,$$

then $\phi(t)=\phi(1)t^p$, $t>0$.

Proof. Put $a:=\mu(A)$. Setting $x:=s\chi_A\in\boldsymbol{S}_+(\Omega,\Sigma,\mu)$ for $s>0$, gives

$$\boldsymbol{p}_\phi[x+(ts^{-1})x]=\boldsymbol{p}_\phi(x)+\boldsymbol{p}_\phi(ts^{-1}x), \quad s,t>0,$$

which, by Remark 1, can be written in the form

$$\phi^{-1}[a\phi(s+t)]=\phi^{-1}[a\phi(s)]+\phi^{-1}[a\phi(t)], \quad s,t>0.$$

Thus, the function $\phi^{-1}\circ(a\phi)$, being additive and positive, is linear.

Consequently, there is an $\alpha>0$, such that

$$\phi(\alpha u)=a\phi(u), \quad u>0.$$

As $a \neq 1$, we also have $\alpha \neq 1$. Write this equation in the form

$$\frac{\phi(\alpha u)}{(\alpha u)^p} = \frac{a}{\alpha^p}\frac{\phi(u)}{u^p}, \qquad u > 0. \tag{7}$$

Assume, for instance, that the limit

$$c := \lim_{u \to 0+} \frac{\phi(u)}{u^p}$$

exists and $c > 0$. Letting $u > 0$ tend to 0 in (7) implies that $a = \alpha^p$, and consequently, from (7) we have

$$\frac{\phi(\alpha u)}{(\alpha u)^p} = \frac{\phi(u)}{u^p}, \qquad u > 0.$$

Hence, for $\gamma : (0, \infty) \mapsto (0, \infty)$, defined by the formula

$$\gamma(u) = \frac{\phi(u)}{u^p}, \qquad u > 0,$$

we get the functional equation

$$\gamma(\alpha u) = \gamma(u), \qquad u > 0, \tag{8}$$

and

$$\lim_{u \to 0+} \gamma(u) = c. \tag{9}$$

Since $\gamma(\alpha^{-1}u) = \gamma(u), u > 0$, we can assume, without any loss of generality that $\alpha \in (0, 1)$. From (8) we have

$$\gamma(u) = \gamma(\alpha^n u), \qquad u > 0,\ n \in \mathbf{N}.$$

Hence, letting $n \to \infty$, by (9) we get $\gamma(u) = c$ for all $u > 0$, and by the definition of γ, $\phi(u) = cu^p$, for all $u > 0$.

In the remaining case the proof is similar.

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