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FUNCTIONAL EQUATIONS MOTIVATED BY THE LAGRANGE'S IDENTITY

Abstract. We solve two functional equations motivated by the following Lagrange's identity:

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 = \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big) - \sum_{1\leqslant i<j\leqslant n} (a_i b_j - a_j b_i)^2,$$

which is valid for every $n \in \mathbb{N}$ and each $a_1, \ldots, a_n, b_1, \ldots, b_n$ from a commutative ring.

The classical Lagrange's identity states that for every $n \in \mathbb{N} = \{1, 2, \ldots\}$ and each a_i, b_i from a commutative ring R, where $i = 1, \ldots, n$, we have:

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 = \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big) - \sum_{1\leqslant i<j\leqslant n} (a_i b_j - a_j b_i)^2 \tag{1}$$

or, if division by 2 is uniquely performable in R:

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 = \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (a_i b_j - a_j b_i)^2. \tag{2}$$

These identities motivate the following two functional equations:

$$f\Big(\sum_{i=1}^{n} a_i b_i\Big) = \Big(\sum_{i=1}^{n} f(a_i)\Big)\Big(\sum_{i=1}^{n} f(b_i)\Big) - \sum_{1\leqslant i<j\leqslant n} f(a_i b_j - a_j b_i) \tag{3}$$

and

$$f\Big(\sum_{i=1}^{n} a_i b_i\Big) = \Big(\sum_{i=1}^{n} f(a_i)\Big)\Big(\sum_{i=1}^{n} f(b_i)\Big) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(a_i b_j - a_j b_i), \tag{4}$$

which can be discussed for an unknown mapping f acting between fields, rings or algebras. Dealing with equation (4) we need to assume additionally

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that the target space of f contains unit element 1 and $\frac{1}{2}$. Observe also that equations (3) and (4) need not to be equivalent, unless f is even and $f(0) = 0$.

It is worth to note that another functional equation related to the Lagrange's identity is already known. The following version of the Euler–Lagrange quadratic functional equation:

$$(5)\qquad Q\Big(\sum_{i=1}^{n} a_i x_i\Big) + \sum_{1\leqslant i<j\leqslant n} Q(a_i x_j - a_j x_i) = \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} Q(x_i)\Big)$$

and its several modifications jointly with the corresponding stability questions have been studied by J. M. Rassias [9], [10], [11], [12], H.-M. Kim, J. M. Rassias and Y.-S. Cho [5], H.-M. Kim and J. M. Rassias [6], M. J. Rassias and J. M. Rassias [13], A. Pietrzyk [8], among others.

The main difference between (3) or (4) and (5) seems to lie in the nonlinearity of (3) and (4). Indeed, if Q_1 and Q_2 solve (5) then for each scalars λ_1, λ_2 the map $\lambda_1 Q_1 + \lambda_2 Q_2$ provides a solution of (5). It is clear that (3) and (4) do not possess an analogical property and therefore one may expect a different behavior of these equations.

The purpose of the present paper is to determine general solutions of (3) and (4). Therefore we provide an answer to the question whether, or to what extent, the Lagrange's identity characterizes the mapping $x \mapsto x^2$ on rings or algebras.

Clearly, if the Lagrange's identity holds for a given $n \geqslant 2$ then a simple substitution $a_n = b_n = 0$ proves the validity of this identity for $n-1$ (on a ring or on an arbitrary structure with adequate operations and the zero element). The converse implication is not straightforward and therefore one may ask if the Lagrange's identity assumed for n and then assumed for $n-1$ are equivalent (in a sense that corresponding functional equations have the same solutions). To settle the question we will solve corresponding functional equations (3) or (4), respectively assuming its validity for a fixed n only.

Observe that if $n = 1$ then (3) reduces to the multiplicative Cauchy equation:

$$(6)\qquad f(a_1 b_1) = f(a_1) f(b_1)$$

whereas (4) reduces to

$$(7)\qquad f(a_1 b_1) = f(a_1) f(b_1) - \frac{1}{2} f(0).$$

The description of solutions of (6) is well known in the literature, see e.g. M. Kuczma [7, pp. 343–350]. Under some mild assumptions equation (7) can be easily solved by a reduction to (6).

Proposition 1. *Assume that R and S are arbitrary rings such that $1 \in S$ and $\frac{1}{2}, \frac{1}{3} \in S$, $f\colon R \to S$ and at least one of the elements $f(0)$ and $\frac{3}{2} - f(0)$ is not a zero divisor in S. Then f is a solution of (7) if and only if either f vanishes at zero and satisfies the multiplicative Cauchy equation (6) or f is constant and equal to $\frac{3}{2}$.*

Proof. Substitute $a_1 = b_1 = 0$ in (7) to get

$$\Big(\frac{3}{2} - f(0)\Big) f(0) = 0.$$

Therefore, either $f(0) = 0$ or $f(0) = \frac{3}{2}$ (by our assumption both elements $f(0)$, $\frac{3}{2} - f(0)$ are not zero divisors simultaneously). In the first case (7) is equivalent to (6). If $f(0) = \frac{3}{2}$ then substitution $b_1 = 0$ gives us

$$\frac{3}{2} = f(0) = f(a_1 \cdot 0) = f(a_1) \cdot \frac{3}{2} - \frac{3}{4},$$

i.e. f is constantly equal to $\frac{3}{2}$.

The converse implication is straightforward. ∎

Now, let us discuss the general case with arbitrarily fixed $n > 1$.

Lemma 1. *Assume that $n > 1$ is an integer, R and S are arbitrary rings such that S is commutative, $1 \in S$ and $\frac{1}{2}, \frac{1}{n}, \frac{1}{n^2-n+2} \in S$, $f\colon R \to S$ is a solution of (3) and at least one of the elements $f(0)$ and $\frac{n^2-n+2}{2n^2} - f(0)$ is not a zero divisor in S. Then either $f(0) = 0$ and f satisfies the multiplicative Cauchy equation (6) or f is constant and equal to $\frac{n^2-n+2}{2n^2}$.*

Proof. Put $a_1 = 0, \ldots, a_n = 0$ and $b_1 = 0, \ldots, b_n = 0$ in (3) to obtain

$$f(0) = n^2 f(0)^2 - \frac{n^2 - n}{2} f(0),$$

i.e.

$$\Big(\frac{n^2 - n + 2}{2} - n^2 f(0)\Big) f(0) = 0.$$

By our assumption, from the above equality it follows that either $f(0) = 0$ or $f(0) = \frac{n^2-n+2}{2n^2}$. If $f(0) = 0$ then (3) applied for $a_2 = \ldots = a_n = 0$ and $b_2 = \ldots = b_n = 0$ implies immediately (6). In the second case, fix arbitrarily $x \in R$ and substitute $a_1 = x, \ldots, a_n = x$ and $b_1 = 0, \ldots, b_n = 0$ in (3). We get

$$f(0) = n^2 f(x) f(0) - \frac{n^2 - n}{2} f(0),$$

which implies the equality $f(x) = \frac{n^2-n+2}{2n^2}$. ∎

In a similar way we obtain an analogous result for equation (4).

Lemma 2. *Assume that $n > 1$ is an integer, R and S are arbitrary rings such that S is commutative, $1 \in S$ and $\frac{1}{2}, \frac{1}{n}, \frac{1}{n^2+2} \in S$, $f\colon R \to S$ is a solution of* (4) *and at least one of the elements $f(0)$ and $\frac{n^2+2}{2n^2} - f(0)$ is not a zero divisor in S. Then either $f(0) = 0$ and f satisfies the multiplicative Cauchy equation* (6) *or f is constant and equal to $\frac{n^2+2}{2n^2}$.*

Proof. Substitution $a_1 = 0, \ldots, a_n = 0$ and $b_1 = 0, \ldots, b_n = 0$ in (4) leads to the equality

$$f(0) = n^2 f(0)^2 - \frac{n^2}{2} f(0),$$

which is equivalent to

$$\Big(\frac{n^2+2}{2} - n^2 f(0)\Big) f(0) = 0.$$

Again, if $f(0) = 0$ then one may substitute $a_2 = \ldots = a_n = 0$ and $b_2 = \ldots = b_n = 0$ in (3) to obtain (6). Further, if $f(0) \neq 0$ then $f(0) = \frac{n^2+2}{2n^2}$. Next, for an arbitrarily fixed $x \in R$ put $a_1 = x, \ldots, a_n = x$ and $b_1 = 0, \ldots, b_n = 0$ in (3). We arrive at

$$f(0) = n^2 f(x) f(0) - \frac{n^2}{2} f(0),$$

which gives us the equality $f(x) = \frac{n^2+2}{2n^2}$. ∎

Remark 1. One may easily check that if one of equations (3) or (4) is satisfied for two different integers n_1, n_2 which are greater than one then each solution of (3) or (4), respectively, is a multiplicative mapping. Indeed, since both mappings

$$\{2, 3, \ldots\} \ni n \mapsto \frac{n^2 - n + 2}{2n^2}, \qquad \{2, 3, \ldots\} \ni n \mapsto \frac{n^2+2}{2n^2}$$

are injective then by Lemma 1 and Lemma 2 each solution of (3) and (4) is multiplicative.

Remark 2. If S is a Banach algebra and $\|f(0)\| < \frac{n^2+2}{2n^2}$ then $\frac{n^2+2}{2n^2} - f(0)$ is invertible in S (and thus not a zero divisor).

Our next step is to provide conditions under which each solution of (3) or (4) which satisfy $f(0) = 0$ is a quadratic mapping. We will need to assume additionally that the ring R contains unit element.

Recall that a map $f\colon R \to S$ (between Abelian groups) is called quadratic if and only if f satisfies:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{8}$$

for each $x, y \in R$. It is well known (see e.g. J. Aczél and J. Dhombres [1, Chapter 11, Proposition 1] or K. Baron and P. Volkmann [2, Proposition]) that if unique division by 2 is possible in R or in S then $f\colon R \to S$ is a quadratic mapping if and only if there exists a bi-additive and symmetric functional $B\colon R \times R \to S$ such that

$$f(x) = B(x,x), \quad x \in R.$$

Lemma 3. *Under assumptions of Lemma* 1, *if* $1 \in R$ *and* $f\colon R \to S$ *is a solution of* (3) *such that* $f(0) = 0$ *then* f *is a quadratic mapping.*

Proof. By Lemma 1 f satisfies (6) and thus in particular

$$f(x) = f(x \cdot 1) = f(x)f(1)$$

for every $x \in R$. Apply (3) for $a_3 = \ldots = a_n = 0$, $b_1 = b_2 = 1$ and $b_3 = \ldots = b_n = 0$ to obtain

$$f(a_1 + a_2) = 2f(a_1) + 2f(a_2) - f(a_1 - a_2)$$

for each $a_1, a_2 \in R$, i.e. f is quadratic. ∎

Lemma 4. *Under assumptions of Lemma* 2, *if* $1 \in R$ *and* $f\colon R \to S$ *is a solution of* (4) *such that* $f(0) = 0$ *then* f *is a quadratic mapping.*

Proof. Similarly as in the proof of Lemma (3) we have that $f(x) = f(x)f(1)$ for every $x \in R$. Equation (4) applied for $a_3 = \ldots = a_n = 0$, $b_1 = b_2 = 1$ and $b_3 = \ldots = b_n = 0$ gives us

$$f(a_1 + a_2) = 2f(a_1) + 2f(a_2) - \frac{1}{2}[f(a_1 - a_2) + f(a_2 - a_1)] \tag{9}$$

for each $a_1, a_2 \in R$. Apply this equality for $a_2 = 0$. We see that

$$f(a_1) = 2f(a_1) - \frac{1}{2}f(a_1) - \frac{1}{2}f(-a_1),$$

i.e. f is even. Therefore, by joining this with (9) we see that f is a quadratic mapping. ∎

Now, we may formulate our main result.

Theorem 1. *Assume that* $n > 1$ *is an integer,* R *is a commutative ring with unit,* S *is a field with characteristic different from* 2 *and from* n *and* $f\colon R \to S$ *is not constant. Then the following conditions* (i), (ii) *and* (iii) *are equivalent:*

(i) *f is a solution of* (3);
(ii) *f is a solution of* (4);
(iii) *there exists an additive and multiplicative functional* $u\colon R \to S$ *such that* $f(x) = u^2(x)$ *for each* $x \in R$.

Proof. Implications (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) follow from the Lagrange's identity written in the form (1) or (2), respectively.

We will show the converse implications. Making an advantage of the similarities of equations (3) and (4) we may prove both implications simultaneously.

Since the target space of f is a field and thus in particular contains no zero divisors and f is not constant then by Lemma 1 if f satisfies (3) then $f(0) = 0$ and it is multiplicative and then by Lemma 3 f is quadratic. Similarly, by Lemmas 2 and 4 if f satisfies (4) then f is multiplicative and quadratic.

Let us denote by $\overline{S}$ the algebraic closure of the field S and for any element $\alpha \in \overline{S}$ let the symbol $S(\alpha)$ stand for the smallest field extension of S which contains α.

We will apply a result of Z. Gajda [3, Theorem] which states that a multiplicative and quadratic mapping $f\colon R \to S$, where S is a field with characteristic different from 2 and R a commutative ring with unit, can be represented in the form:

$$f(x) = u(x) \cdot v(x), \quad x \in R,$$

where both $u\colon R \to S(\alpha)$ and $v\colon R \to S(\alpha)$ are additive and multiplicative functionals such that:

$$u(x) + v(x) \in S, \qquad u(x) - v(x) \in \alpha S$$

and $\alpha \in \overline{S}$ is an element which satisfies $\alpha^2 \in S$. Moreover, we may assume that $u(1) = v(1) = 1$ (clearly, other possibility for multiplicative functionals is $u(1) = 0$ or $v(1) = 0$ which leads to $u = 0$ or $v = 0$ and, consequently, to $f = 0$).

Now, we will join the representation $f(x) = u(x) \cdot v(x)$ with (3) or (4), respectively. Fix arbitrarily an $x \in R$ and let us put $a_1 = x$, $a_2 = 1$ and $a_3 = \ldots = a_n = 0$ in case $n \geqslant 3$, $b_1 = x$, $b_2 = 1$ and $b_3 = \ldots = b_n = 0$ if $n \geqslant 3$ in (3) or (4), respectively. We have:

$$\begin{aligned} f\Big(\sum_{i=1}^{n} a_i b_i\Big) &= f(x^2+1) = u(x^2+1)v(x^2+1) \\ &= u^2(x)v^2(x) + u^2(x) + v^2(x) + 1, \\ \Big(\sum_{i=1}^{n} f(a_i)\Big)\Big(\sum_{i=1}^{n} f(b_i)\Big) &= [f(x)+f(1)]^2 = u^2(x)v^2(x) + 2u(x)v(x) + 1, \end{aligned}$$

and

$$\sum_{1 \leqslant i < j \leqslant n} f(a_i b_j - a_j b_i) = f(x \cdot 1 - 1 \cdot x) = f(0) = 0,$$

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(a_ib_j - a_jb_i) = \frac{1}{2}[f(x^2 - x^2) + f(x\cdot 1 - 1\cdot x) + f(1\cdot x - x\cdot 1) + f(1^2 - 1^2)] = 0.$$

Therefore, both (3) and (4) turn into

$$u^2(x)v^2(x) + u^2(x) + v^2(x) + 1 = u^2(x)v^2(x) + 2u(x)v(x) + 1,$$

which immediately gives us

$$[u(x) - v(x)]^2 = 0.$$

Therefore, we have proved the equality $u = v$, which means that

$$f(x) = u^2(x), \quad x \in R.$$

Finally, from the equality $u = v$ we easily deduce that $\alpha \in S$ and thus the range of u is contained in the field S. ■

Remark 3. Theorem of Z. Gajda [3], used in the proof of our Theorem 1, is a generalization of a result of C. Hammer and P. Volkmann from [4]. They have proved that a mapping $f\colon \mathbb{R} \to \mathbb{R}$ which is multiplicative and quadratic can be written as

$$f(x) = (\Re w(x))^2 + (\Im w(x))^2,$$

with additive and multiplicative $w\colon \mathbb{C} \to \mathbb{C}$. However, in case of real-to-real solutions of (3) or (4) we can easily calculate that the function

$$f(x) = x^2, \quad x \in \mathbb{R}$$

is the only nonconstant solution of (3) or (4), since it is well known that the only nonzero additive multiplicative real-to-real functional is the identity mapping.

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