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Citation style: Czerwik Stefan. (1976). Generalization of Edelstein's fixed point theorem. "Demonstratio Mathematica" (Vol. 9, nr 2 (1976) s. 281-285), doi 10.1515/dema-1976-0215



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GENERALIZATION OF EDELSTEIN'S FIXED POINT THEOREM

1. Introduction

Let (X_i, d_i) , $i = 1, \dots, n$ be metric spaces. Let T_i , $i=1, \dots, n$ be transformations mapping of $B \stackrel{\text{def}}{=} X_1 \times \dots \times X_n$ into X_i . For any positive number a we define (cf. also [3])

$$Z_a = \left\{ (x_1, \dots, x_n) \in B : d_i [x_i, T_i(x_1, \dots, x_n)] \leq a, i=1, \dots, n \right\}.$$

In [1] the following fixed point theorem has been proved, generalizing the Banach principle for contraction maps (cf. [4]):

Let E be a metric space and T an operator which transforms E into itself. Suppose that $d[T(x), T(y)] < d(x, y)$, $x \neq y$, $x, y \in E$. Assume that there exists $x \in E$ such that the sequence at iterates $\{T^m(x)\}$ contains a subsequence $\{T^{m_F}(x)\}$ convergent to a point $u \in E$. Then u is a unique fixed point of T .

The purpose of the present paper is to prove (using the notation of the sets Z_a) a fixed point theorem which generalizes the Edelstein's theorem and the result in [5].

2. Edelstein's fixed point theorem

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, $x_i, y_i \in X_i$, $i=1, \dots, n$, and let

$$\Delta = \left\{ (x, y) \in B \times B : x_i = y_i, i=1, \dots, n \right\}.$$

We shall prove the following theorem.

T h e o r e m. Let (X_i, d_i) , $i=1, \dots, n$ be metric spaces. Suppose that the transformations $T_i : B \rightarrow X_i$, $i=1, \dots, n$, fulfil the following conditions

$$(1) \quad d_i [T_i(x), T_i(y)] < \sum_{k=1}^n a_{ik} d_k(x_k, y_k) \quad \text{in } Y = B \times B - \Delta,$$

$$(2) \quad |\lambda_i| \leq 1, \quad i=1, \dots, n,$$

where $a_{ik} > 0$, $i, k = 1, \dots, n$ and λ_i , $i=1, \dots, n$ are the characteristic roots of the matrix (a_{ik}) , $i, k = 1, \dots, n$. Assume that there exists a point $u = (u_1, \dots, u_n) \in B$ such that the sequence at iterates $\{T_i^m(u)\}$ contains a subsequence $\{T_i^m(u)\}$ convergent to $z_i \in X_i$, $i=1, \dots, n$. Then $z = (z_1, \dots, z_n)$ is a unique fixed point of the system of equations

$$(3) \quad x_i = T_i(x), \quad i=1, \dots, n.$$

P r o o f. From (2) and Perron's theorem ([2], p.354), it follows that there exist positive numbers q_i , $i=1, \dots, n$ such that

$$(4) \quad \sum_{k=1}^n a_{ik} q_k \leq q_i, \quad i=1, \dots, n.$$

Suppose that there exists an integer v , $1 \leq v \leq n$ such that $z_v \neq T_v(z)$. We define the functions

$$f_i(x, y) = \frac{d_i [T_i(x), T_i(y)]}{\sum_{k=1}^n a_{ik} d_k(x_k, y_k)}, \quad i=1, \dots, n, \quad (x, y) \in Y = B \times B - \Delta.$$

The functions f_i , $i=1, \dots, n$ are continuous in Y . We see that $f_i(z, T(z)) < Q < 1$ ($T(z) = (T_1(z), \dots, T_n(z))$). Hence there exists a neighbourhood U of $(z, T(z))$ such that for $(x, y) \in U$ we have $f_i(x, y) < Q$, $i=1, \dots, n$. Also there exist neighbourhoods U_1 and U_2 of z and $T(z)$ respec-

tively, such that $U_1 \times U_2 \subset U$ and for $r \geq s$ we have $T_1^{m,r}(u) \in U_1$ and $T_1^{m,r+1}(u) \in U_2$. Consequently, for $r \geq s$ and $i = 1, \dots, n$ we obtain

$$(5) \quad d_i \left[T_1^{m,r+1}(u), T_1^{m,r+2}(u) \right] < Q \sum_{k=1}^n a_{ik} d_k \left[T_k^{m,r}(u), T_k^{m,r+1}(u) \right].$$

Since the system of inequalities (4) is homogeneous, we may assume that

$$(6) \quad q_i \geq d_i \left[T_1^{m,s}(u), T_1^{m,s+1}(u) \right], \quad i = 1, \dots, n.$$

From (1), (5), (6) and (4) we obtain

$$(7) \quad d_i \left[T_1^{m,s+r}(u), T_1^{m,s+r+1}(u) \right] < Q^r q_i, \quad r=1,2,\dots, i=1,\dots,n.$$

Let now a_m be a decreasing sequence of positive numbers tending to zero. We denote

$$Z_m = \left\{ y \in B : d_i(y_i, T_1(y)) \leq a_m, \quad i = 1, \dots, n \right\}.$$

Since $0 \leq Q < 1$, (7) implies that the sets Z_m are non empty. Let m be a fixed positive integer and consider the corresponding set Z_m . We have

$$\begin{aligned} d_i \left[z_i, T_1(z) \right] &\leq d_i \left[z_i, T_1^{m,r}(u) \right] + d_i \left[T_1^{m,r}(u), T_1^{m,r+1}(u) \right] + \\ &\quad + d_i \left[T_1^{m,r+1}(u), T_1(z) \right]. \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Since

$$T_1^{m,r}(u) \xrightarrow{r \rightarrow \infty} z_i, \quad T_1^{m,r+1}(u) \xrightarrow{r \rightarrow \infty} T_1(z), \quad i = 1, \dots, n$$

and

$$\left(T_1^m(u), \dots, T_n^m(u) \right) \in Z_m$$

for all sufficiently large values of r , there exists a integer N such that for $r > N$

$$d_i[z_i, T_i(z)] \leq 2\varepsilon + a_m, \quad i = 1, \dots, n.$$

Since ε is arbitrary, it follows that $d_i[z_i, T_i(z)] \leq a_m$ and $z = (z_1, \dots, z_n) \in Z_m$. Therefore $z \in Z_m$ for every integer m and consequently $z_i = T_i(z_1, \dots, z_n)$, $i = 1, \dots, n$. This contradiction proves that $z = (z_1, \dots, z_n)$ is a fixed point of the system of equations (3).

Now we shall prove that $z = (z_1, \dots, z_n)$ is a unique fixed point of the system of equations (3). Suppose that there exists $b = (b_1, \dots, b_n)$, $b \neq z$ such that $b_i = T_i(b)$, $i = 1, \dots, n$. We see that

$$f_i(b, z) < Q < 1, \quad i = 1, \dots, n.$$

We can prove (as in the preceding case) that for $m > N$ and $r = 1, 2, \dots$, we have

$$d_i(b_i, z_i) \leq d_i[T_i^{m+r}(b), T_i^{m+r}(z)] \leq Q^r \max_i(d_i[T_i^m(b), T_i^m(z)]).$$

Passing to the limit as $r \rightarrow \infty$, we obtain

$$d_i(b_i, z_i) = 0, \quad i = 1, \dots, n,$$

and $b = z$. This contradiction proves the theorem.

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Received July 5, 1975.