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IRREGULAR SOLUTIONS OF THE FEIGENBAUM FUNCTIONAL EQUATION

Abstract. We describe the structure of orbits generated by two commuting bijections and using this description we construct irregular solutions of the Feigenbaum functional equation:

$$\varphi(\varphi(\lambda x)) = \lambda\varphi(x) = 0$$

and its generalizations:

$$\varphi^2(x) = g(\varphi(f(x))).$$

The graph of such a solution almost cover the plane in the sense of measure and topology.

1. Introduction

Let X and Y be two sets and $\mathcal{R}$ be a family of subsets of $X \times Y$. We say that $\varphi : X \to Y$ has a *big graph* with respect to $\mathcal{R}$ if its graph $\mathrm{Gr}\varphi$ meets every set of $\mathcal{R}$. We are interested in finding conditions under which functional equations of the form

$$\varphi^2(x) = g(\varphi(f(x))) \tag{1}$$

have a solution with big graph with respect to a sufficiently large family. (More exactly, we would like to obtain such a solution under conditions which are automatically satisfied in the case of the famous Feigenbaum equation

$$\varphi(\varphi(\lambda x)) + \lambda\varphi(x) = 0. \tag{2}$$

The idea of constructing solutions with big graphs go back to F. B. Jones [8] (see also [11, Ch.12, §4], [1]). Since [9] by P. Kahlig and J. Smítal it becomes known in itèrative functional equtions theory (cf. [2]). Unfortunetly, concerning the Feigenbaum equation, we can apply to it only [3, Corollary 1] and only in the case where $\lambda = 1$. It is the aim of this paper to fill this gap.

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The main part of the paper we supply by topological and measure-theoretical properties of functions with big graph taken from [3]. It shows in particular that we are able to get a solution of (1) with the graph almost covering $X \times Y$ in the sense of measure and topology.

The results have been presented on The Twenty-Sixth Summer Symposium in Real Analysis and appeared (without proofs) in [4].

2. Main result

Let X be a nonempty set and suppose that $f: X \to X$ and $g: X \to X$ are commuting bijections (one-to-one and onto). For every $x \in X$ denote by $C(x)$ the orbit of the point x generated by functions f and g, i.e. the equivalence class containing x of the relation $\sim$ on X defined by

$$x \sim y \Longleftrightarrow y = f^m g^n(x) \text{ for some } m, n \in \mathbb{Z}.$$

Clearly

$$C(x) = \{f^m g^n(x) : m, n \in \mathbb{Z}\}.$$

The following obvious remarks slightly explain how to construct a solution of (1).

Remark 1. For every $x \in X$ the function $\varphi : C(x) \to C(x)$ given by $\varphi = f \circ g|_{C(x)}$ is a solution of (1) and commutes with f and g.

Remark 2. Assume $\varphi : X \to X$ is a solution of (1) which commutes with f and g. Then:

(i) if $y \in \varphi(X)$ then $\varphi(y) = f(g(y))$;
(ii) for every $x \in X$ we have $\varphi(C(x)) = C(\varphi(x))$.

Recall some notions. We say that an $x \in X$ is a *periodic* point of f with *period* p (being a positive integer) iff

$$f^p(x) = x, \; f^k(x) \neq x \text{ for } k = 1, \dots, p-1.$$

The set of all such points will be denoted by $\mathrm{Per}(f,p)$ and we put

$$\mathrm{Per} f = \bigcup_{p=1}^{\infty} \mathrm{Per}(f,p).$$

Below we clasify all the possible types of orbits generated by two commuting bijections.

Definition. Let $x \in X$.

(i) The orbit $C(x)$ is of the type $(0,0)$ iff

$$f^k(x) \neq g^l(x) \text{ for } k, l \in \mathbb{Z}, |k| + |l| \neq 0.$$

(ii) If m is a positive integer then the orbit $C(x)$ is of the type $(m, 0)$ iff

$$x \in \mathrm{Per}(f,m) \text{ and } f^k(x) \neq g^l(x) \text{ for } k, l \in \mathbb{Z}, l \neq 0,$$

and it is of the type $(0, m)$ iff

$$x \notin \mathrm{Per}(f) \text{ and } g^m(x) = f^k(x) \text{ for some } k \in \mathbb{Z}$$
$$\text{and } g^l(x) \neq f^k(x), \text{ for } 0 < l < m, k \in \mathbb{Z}.$$

(iii) If m and n are positive integers then the orbit $C(x)$ is of the type (m, n) iff

$$x \in \mathrm{Per}(f, m),\ g^n(x) = f^k(x) \text{ for some } k \in \mathbb{Z},$$
$$\text{and } g^l(x) \neq f^k(x) \text{ for } 0 < l < n, k \in \mathbb{Z}.$$

For nonnegative integers m, n let $L_{m,n}$ denote the (cardinal) number of all orbits of the type (m, n).

Note that in the case of the Feigenbaum equation on reals we have $f(x) = \frac{1}{\lambda}x$ and $g(x) = -\lambda x$ whence

$$C(x) = \{\lambda^n x,\ -\lambda^n x :\ n \in \mathbb{Z}\}.$$

Hence $C(0)$ is of the type $(1, 1)$ and for every λ different from 1 and -1 and for every $x \neq 0$ the orbit $C(x)$ is of the type $(0, 2)$. Consequently

$$L_{1,1} = 1,\ L_{0,2} = \mathfrak{c}$$

and $L_{m,n} = 0$ for (m, n) different from $(1, 1)$ and $(0, 2)$.

For any set $R \subset X \times X$ and $x \in X$ we denote by R_x the vertical section of R, i.e. the set $\{y \in X : (x, y) \in R\}$. The following is our main result.

THEOREM 1. *Assume X is uncountable and let f and g be commuting bijections of X such that there are nonnegative integers m_0, n_0 with*

$$\mathrm{card} X = L_{m_0,n_0} \tag{3}$$

and

$$\sum_{(m,n)\neq(m_0,n_0)} L_{m,n} < \mathrm{card} X. \tag{4}$$

If $\mathcal{R}$ is a family of subset of $X \times X$ such that

$$\mathrm{card} \mathcal{R} \leq \mathrm{card} X \tag{5}$$

and

$$\mathrm{card}\{x \in X :\ \mathrm{card} R_x = \mathrm{card} X\} = \mathrm{card} X \ \text{ for } R \in \mathcal{R} \tag{6}$$

then there exists a solution $\varphi : X \to X$ of (1) which commutes with f and g and has a big graph with respect to $\mathcal{R}$.

Proof. Let

$$A = \{x \in X :\ \ C(x) \text{ is of the type } (m_0, n_0)\}.$$

According to (3) we have

$$\mathrm{card} A = \mathrm{card} X. \tag{7}$$

Hence and from (4) for

$$A_{-1} = X \setminus A$$

we get

(8) $$\mathrm{card} A_{-1} < \mathrm{card} X.$$

Define $\varphi_{-1} \colon A_{-1} \to A_{-1}$ by $\varphi_{-1} = f \circ g|_{A_{-1}}$. Then (cf. Remark 1)

(9) $$\varphi_{-1}^2 = g \circ \varphi_{-1} \circ f|_{A_{-1}}.$$

Now we start to define a solution of (1) on the set A. Let γ be the smallest ordinal such that its cardinal equals that of $\mathcal{R}$ and let $(R_\alpha : \alpha < \gamma)$ be a one-to-one transfinite sequence of all the elements of $\mathcal{R}$. We shall define a sequence $(A_\alpha \colon \alpha < \gamma)$ of countable subsets of A and a sequence $(\varphi_\alpha \colon \alpha < \gamma)$ of functions $\varphi_\alpha \colon A_\alpha \to A_\alpha$ such that for every $\alpha < \gamma$ the following conditions (10)–(13) hold:

(10) $$f(A_\alpha) = A_\alpha, \; g(A_\alpha) = A_\alpha,$$

(11) $$\varphi_\alpha^2 = g \circ \varphi_\alpha \circ f|_{A_\alpha}, \; \varphi_\alpha \circ f|_{A_\alpha} = f|_{A_\alpha} \circ \varphi_\alpha, \; \varphi_\alpha \circ g|_{A_\alpha} = g|_{A_\alpha} \circ \varphi_\alpha,$$

(12) $$A_\beta \cap A_\alpha = \emptyset \quad \text{for} \quad \beta < \alpha,$$

(13) $$\mathrm{Gr}\varphi_\alpha \cap R_\alpha \neq \emptyset.$$

Suppose $\alpha < \gamma$ and that we have already defined suitable A_β's and φ_β's for every $\beta < \alpha$. According to (6)–(8) we have

$$\mathrm{card}\{x \in A : \; \mathrm{card}(R_\alpha)_x = \mathrm{card} X\} = \mathrm{card} A$$

which allows us to fix an $x \in A \setminus \bigcup_{\beta<\alpha} A_\beta$ such that

$$\mathrm{card}(R_\alpha)_x = \mathrm{card} X.$$

Taking (8) into account we see that $\mathrm{card} A \cap (R_\alpha)_x = \mathrm{card} X$ and we can find

(14) $$y \in (A \cap (R_\alpha)_x) \setminus \Big(\bigcup_{\beta<\alpha} A_\beta \cup C(x) \Big).$$

Put $A_\alpha = C(x) \cup C(y)$ and define $\varphi_\alpha \colon A_\alpha \to A_\alpha$ by

$$\varphi_\alpha|_{C(y)} = f \circ g|_{C(y)}$$

and

(15) $$\varphi(f^m g^n(x)) = f^m g^n(y) \text{ for } m, n \in \mathbb{Z}.$$

Since orbits $C(x)$ and $C(y)$ are of the same type (m_0, n_0), the function φ_α is well defined. It is easy to see that (10) - (12) hold. According to (15) we have $\varphi_\alpha(x) = y$ which jointly with (14) gives (13). The transfinite induction is completed.

With

$$A_{-2} = A \setminus \bigcup_{\alpha<\gamma} A_\alpha$$

we define a function $\varphi_{-2} : A_{-2} \to A_{-2}$ by

$$\varphi_{-2} = f \circ g|_{A_{-2}}.$$

Then

$$\varphi_{-2}^2 = g \circ \varphi_{-2} \circ f|_{A_{-2}}. \tag{16}$$

Now $X = A_{-1} \cup A_{-2} \cup \bigcup_{\alpha<\gamma} A_\alpha$ with disjoint summands which allows us to define a function $\varphi\colon X \to X$ by

$$\varphi = \varphi_{-1} \cup \varphi_{-2} \cup \bigcup_{\alpha<\gamma} \varphi_\alpha.$$

According to (9), (11) and (16) it is a solution of (1). Due to (13), which holds for every $\alpha < \gamma$, φ has a big graph with respect to $\mathcal{R}$. Since every φ_δ commutes both with f and g, the same concerns φ. ∎

COROLLARY. *Assume λ is a nonzero real number. If $\mathcal{R}$ is a family of subsets of $\mathbb{R} \times \mathbb{R}$ such that* $\operatorname{card}\mathcal{R} \le \mathfrak{c}$ *and*

$$\operatorname{card}\{x \in \mathbb{R}\colon \operatorname{card}R_x = \mathfrak{c}\} = \mathfrak{c} \quad \textit{for } R \in \mathcal{R}$$

then there exists an even solution $\varphi\colon \mathbb{R} \to \mathbb{R}$ of (2) *which has a big graph with respect to $\mathcal{R}$ and such that*

$$\varphi(\lambda x) = \lambda\varphi(x) \quad \textit{for } x \in \mathbb{R}.$$

3. Properties of functions with big graph

Given a topological space X, consider the family

$$\{R \in \mathcal{B}(X \times X)\colon \{x \in X\colon R_x \text{ is uncountable}\} \text{ is uncountable}\} \tag{17}$$

where $\mathcal{B}(X \times X)$ denotes the σ-algebra of all Borel subsets of $X \times X$.

The following remark is a consequence of the theorem of Mazurkiewicz–Sierpiński ([10, 29.19]), the theorem of Souslin ([12, p. 437], [10, 29.1]) and the fact that there are not more than $\mathfrak{c}$ many Borel sets in a Polish space.

REMARK 3. If X is an uncountable Polish space, then the family $\mathcal{R}$ defined by (17) satisfies the requirements (5) and (6) of Theorem.

The following observation shows that if a function $\varphi : X \to X$ has a big graph with respect to the family (17), then its graph is big from the topological point of view.

PROPOSITION 1. *Assume T_1-space X has a countable base and has no isolated point. If $\varphi : X \to X$ has a big graph with respect to the family* (17) *then the set $(X \times X) \setminus \operatorname{Gr}\varphi$ contains no subset of $X \times X$ of second category having the property of Baire.*

Proof. Assume $(X\times X)\setminus\mathrm{Gr}\varphi$ contains a set F of second category having the property of Baire. Let G be a second category $\mathcal{G}_\delta$ subset of $X\times X$ contained in F. Clearly the set G does not belong to the family (17). Consider the sets

$$\bigcup\{\{x\}\times G_x\colon\ G_x \text{ is countable}\},\quad \bigcup\{\{x\}\times G_x\colon\ G_x \text{ is uncountable}\}$$

summing up to G. Since G is not in (17), the second one is a countable sum of Borel sets. Consequently both these sets are Borel. Since all the sections of the first one are of first category, the set itself is of first category according to the Kuratowski-Ulam Theorem (see [13, Theorem 15.4], [10, 8.41]). Hence $\{x\in X : G_x \text{ is uncountable}\}$ is uncountable, i.e., G belongs to the family (17), a contradiction. ∎

Making use of the Fubini Theorem, instead of that of Kuratowski-Ulam, (and the fact that $\mathcal{B}(X\times X)$ coincides with the product σ-algebra $\mathcal{B}(X)\times\mathcal{B}(X)$ if X has a countable base) we obtain the following measure-theoretic analogue of Proposition 1.

PROPOSITION 2. *Assume X is a T_1-space with a countable base. Let μ and ν be σ-finite Borel measures on X vanishing on all the singletons. If $\varphi\colon X\to X$ has a big graph with respect to the family* (17) *then the set $(X\times X)\setminus\mathrm{Gr}\varphi$ contains no Borel subset of $X\times X$ of positive product measure $\mu\times\nu$.*

In other words $(\mu\times\nu)_*(X\times X\setminus\mathrm{Gr}\varphi)=0$ and, consequently, $(\mu\times\nu)^*(B\cap\mathrm{Gr}\varphi)=(\mu\times\nu)(B)$ for every $B\in\mathcal{B}(X\times X)$. Here λ_* and λ^* denote inner and outer measures, respectively, generated by a Borel measure λ; cf. [7, Sec. 14].

It is worthwhile to mention that if a Polish space has no isolated point then there are lot of Borel measures on it vanishing on all the singletons [14, p. 55].

Assume now that X is an abelian Polish group. Following J. P. R. Christensen [5], [6, p. 115] we say that a Borel subset R of $X\times Y$ is a Haar zero set if there exists a probability measure λ on $\mathcal{B}(X\times X)$ such that $\lambda(R+z)=0$ for every $z\in X\times X$. Using a version of the Fubini Theorem for Haar zero sets established by J. P. R. Christensen on pp. 259–260 of [5], we have the following.

PROPOSITION 3. *Assume X is locally compact abelian Polish group without isolated points. If $\varphi\colon X\to X$ has a big graph with respect to the family* (17) *then the set $(X\times X)\setminus\mathrm{Gr}\varphi$ contains no Borel subset of $X\times X$ being not a Haar zero set.*

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