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NOTE ON CONVEX FUNCTIONS BOUNDED ON REGULAR HYPERSURFACES

Dedicated to Professor S. Gołąb
on the occasion of his 70^{th} birthday

1. Let R^n denote the Cartesian product of n copies of the space R of real numbers. A real valued function φ defined on an open and convex subset Δ of R^n is called convex iff the inequality (of Jensen)

$$\varphi\left(\frac{x+y}{2}\right) \leq \frac{\varphi(x)+\varphi(y)}{2}$$

holds for every pair $(x,y) \in \Delta \times \Delta$.

M.Kuczma and the present author have introduced in [2] some set classes relevant to the notion of convex function. In particular $\mathcal{A}_n$ is there defined to be the family of all sets $T \subset R^n$ such that every convex function upper-bounded on T is continuous. The question whether a subset T of R^n is a member of $\mathcal{A}_n$ is of interest and has been a subject of many papers (a survey of results and references are given in [3]). One of the classical results says that $\mathcal{A}_n$ contains all sets in R^n with positive inner n-dimensional Lebesgue measure. However, it is also well known that some sets of measure zero turn out to be members of $\mathcal{A}_n$. Recently M.Kuczma has proved that the graph of an arbitrary nonlinear real valued and continuous function defined on an interval belongs to $\mathcal{A}_2$ ([4], theorem

4.2). A simple counter-example is there also given to show that the assumption of nonlinearity is essential.

The quoted result suggests the investigation which hypersurfaces in R^n are members of $\mathcal{A}_n$. The purpose of the present paper is to give a partial answer to that question.

2. We are going to prove the following

T h e o r e m. Every (n-1)-dimensional regular hypersurface H in R^n, not contained in an (n-1)-dimensional hyperplane, is a member of $\mathcal{A}_n$, whenever $n \geq 2$[1].

P r o o f: Since every k-dimensional regular hypersurface in R^n, $1 \leq k < n$, yields locally the graph of a map $f\colon G \to R^{n-k}$ where G is an open and connected subset of R^k, we can confine ourselves to the case of hypersurfaces being graphs. Thus we assume that

$$H = \{(x, f(x)) : x \in G\} \tag{1}$$

where G denotes an open subset of R^{n-1} and f is a nonlinear real-valued function of class C^1, defined on G.

Let a_o be a point in G such that f is nonlinear in any neighbourhood of a_o. Without loss of generality we can assume that $a_o = 0$ as well as $f(0) = 0$ (cf. [4], theorem 1.2). Suppose that $K(0,\varepsilon)$ is an open ball centered at 0 with a radius $\varepsilon > 0$ such that $3\,K(0,\varepsilon) \subset G$[2].

There exists a point $x_1 \in K(0,\varepsilon) \setminus \{0\}$ such that $f'(0)\cdot x_1 - f(x_1)$ is different from zero (otherwise f would have to be linear). The continuity of the function

$$g(x) = f'(0)\cdot x - f(x)$$

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1) H is called to be a k-dimensional regular hypersurface in R^n $1 \leq k \leq n$, if every point of H has a neighbourhood (in the induced topology) diffeomorphic with an open set in R^k.

2) For $A, B \subset R^k$ and α, β - real numbers we write $\alpha A + \beta B \stackrel{df}{=} \{x \in R^k : x = \alpha a + \beta b,\ a \in A,\ b \in B\}$. Similarly $(\alpha,\beta)\, a \stackrel{df}{=} \{x \in R : x = \gamma a,\ \gamma \in (\alpha,\beta)\}$, $a \in R^k$.

implies the existence of a $\delta_1 > 0$ such that $g(x) \neq 0$ for $x \in K(x_1, \delta_1) \subset K(0,\varepsilon)$ whereas $0 \notin \mathrm{cl}\, K(x_1, \delta_1)$. The case where there exists a point $\bar{x} \in K(0,\varepsilon)\setminus\{0\}$ such that f is linear in a neighbourhood of $\bar{x}$ will be considered separately. Now we can assume that f is nonlinear in any neighbourhood of any point $x \in K(0,\varepsilon)$. Then there exist a $\delta > 0$ and a point x_0 such that $K(x_0,\delta) \subset K(x_1, \delta_1)$ and $f'(x) \neq f'(0)$ for $x \in K(x_0, \delta)$. Finally, for every x from $K(x_0, \delta)$ the following conditions are satisfied:

(i) $f'(x) \neq f'(0)$,

(ii) $g(x) \neq 0$,

(iii) f is nonlinear in any neighbourhood of x.

Let us denote by ψ the function $x \longrightarrow (x,f(x))$, $x \in G$, and put $U = \psi(K(x_0, \delta))$. Evidently $R^n \ni 0 = \psi(0) \notin \psi(\mathrm{cl}\, K(x_0, \delta))$. Moreover, (iii) implies that U is not contained in any (n-1)-dimensional hyperplane. Thus we are able to construct the "cone" S with the vertex $0 = \psi(0)$ and the "base" U, i.e. the set

$$S = \bigcup_{p \in U} \bigcup_{\lambda \in [0,1]} \{\lambda p\}.$$

The interior of S is non-void in the topology of R^n. Now, we are going to show that

$$(2) \qquad S \subset Q(H) ,$$

where Q(H) is the Q-convex hull of H, i.e. the set of all finite linear combinations $\sum_{i=1}^{m} \alpha_i x_i$, where m is any positive integer, each x_i is in H, each α_i is nonnegative and belongs to the set Q of rationals, and $\sum_{i=1}^{m} \alpha_i = 1$. It is straightforward to verify that a set T in R^n and its Q-convex hull Q(T) simultaneously belong or do not belong to $\mathcal{A}_n$ (cf. for instance [1]).

To check the above inclusion (2) let us define the function $F_a : K(0,\varepsilon) \times (0,1) \longrightarrow R$ by the formula

(3) $$F_a(x, \lambda) = f(x) + f(2\lambda a - x) - 2\lambda f(a)$$

where $x \in K(0,\varepsilon)$, $\lambda \in (0,1)$ and a is an arbitrarily fixed point of $K(x_0,\delta)$. Since for $a, x \in K(0,\varepsilon)$ we have $2\lambda a - x \in 2K(0,\varepsilon) + K(0,\varepsilon) = 3K(0,\varepsilon) \subset G$, the definition of F_a is correct.

Observe that

$$F_a(a, \tfrac{1}{2}) = f(a) + f(0) - f(a) = 0 ,$$

whereas

$$\frac{\partial F_a}{\partial \lambda}(a, \tfrac{1}{2}) = 2\left[f'(2\lambda a - x)\cdot a - f(a)\right]\Bigg|_{(a, \frac{1}{2})} = 2\,g(a) \neq 0$$

in view of (ii).

Applying the implicit function theorem we infer that in a neighbourhood $K(a,\rho) \subset K(x_0,\delta)$, $\rho > 0$, of the point a there exists a continuous function $\lambda = \lambda(x)$ taking values in $(0,1)$ such that $\lambda(a) = \frac{1}{2}$ and

(4) $$F_a(x, \lambda(x)) \equiv 0 \qquad \text{for} \qquad x \in K(a,\rho).$$

$\lambda(x) \not\equiv$ const. In fact, if $\lambda(x) \equiv \frac{1}{2}$ for $x \in K(a,\rho)$, then (3) and (4) imply

$$f(x) + f(a - x) - f(a) \equiv 0 \qquad \text{for} \qquad x \in K(a,\rho),$$

or equivalently (for $x \neq a$)

(5) $$\frac{f(x) - f(a) - f'(a)\cdot(x - a)}{|x - a|} + \frac{f(a - x) - f'(0)\cdot(a - x)}{|a - x|} = (f'(0) - f'(a)) \cdot \frac{x - a}{|x - a|}.$$

However this is impossible, since in virtue of the differentiability of f at a and 0 respectively, letting x tend to a one would obtain that the left-hand side of (5) converges to zero. This implies $f'(0) = f'(a)$ which is incompatible with (i).

Thus λ establishes a nonconstant continuous map of $K(a,\varrho)$ into $(0,1)$ and hence the image of $K(a,\varrho)$ by λ must contain an open interval $(\alpha,\beta) \subset (0,1)$. Let us take an arbitrary point $x \in K \overset{\mathrm{df}}{=} \lambda^{-1}((\alpha,\beta))$ and put $y(x) \overset{\mathrm{df}}{=} 2\lambda(x)a - x \in G$. The identity (4) assumes now the following form

$$f(x) + f\big(y(x)\big) - 2\lambda(x)\, f(a) \equiv 0 \qquad \text{for} \qquad x \in K ,$$

or equivalently

$$(6) \qquad \lambda(x)\, f(a) = \frac{f(x) + f\big(y(x)\big)}{2} \qquad \text{for} \qquad x \in K .$$

By the definition of y we have also

$$(7) \qquad \lambda(x)\, a = \frac{x + y(x)}{2} \qquad \text{for} \qquad x \in K .$$

(6) and (7) can be jointly written in the form

$$(8) \qquad \lambda(x)\, \psi(a) = \frac{1}{2}\Big(\psi(x) + \psi\big(y(x)\big)\Big) \qquad \text{for} \qquad x \in K .$$

The last relation (8) says that the open segment $(\alpha\psi(a);\ \beta\psi(a)) = (\alpha,\beta)\,\psi(a)$ is contained in $\frac{1}{2}(H + H) \subset Q(H)$. Since also $0 = \psi(0)$ and $\psi(a)$ are elements of $Q(H)$, the whole closed segment $[0\,;\psi(a)]$ turns out to be a subset of $Q(H)$. In fact, $[0,\beta)\,\psi(a)$ and $(\alpha,1]\,\psi(a)$ yield the Q-convex hulls of the sets $\{0\} \cup (\alpha,\beta)\,\psi(a)$ and $(\alpha,\beta)\,\psi(a) \cup \{\psi(a)\}$, respectively.

Because of the free choice of a from $K(x_0,\delta)$ (which is, of course, equivalent to the free choice of a point p from U) the inclusion (2) has been proved. Finally, the interior of $Q(H)$ is non-void which implies that $Q(H)$, and hence H, belongs to $\mathcal{A}_n$.

In order to complete the proof of our assertion it remains yet to consider the case $f(x) = c \cdot x + \gamma$ for $x \in K(\bar{x}, \varrho) \subset$

$\subset K(0,\varepsilon)$, $\bar{x} \neq 0$, $\varrho > 0$. Since f is nonlinear in any neighbourhood of 0, one may find a point $z_0 \in K(0,\varepsilon)$, $z_0 \notin \mathrm{cl}\, K(\bar{x},\varrho)$, such that

$$f(z_0) \neq c \cdot z_0 + \gamma \quad \text{and} \quad f'(z_0) \neq c .$$

Let us fix arbitrarily a point $a \in K(\bar{x},\varrho)$ and define the function $F_a: K(0,\varepsilon) \times (0,1) \longrightarrow R$ by the formula

$$F_a(x,\lambda) = f(x)+f\Big(2\big(\lambda z_0 + (1-\lambda)a\big) - x\Big) - 2\lambda f(z_0) - 2(1-\lambda) f(a)$$

Then

$$F_a(z_0, \tfrac{1}{2}) = 0 \quad \text{and} \quad \frac{\partial F_a}{\partial \lambda}(z_0, \tfrac{1}{2}) = 2\big[c \cdot z_0 + \gamma - f(z_0)\big] \neq 0.$$

By the same argument as in the previous case we derive the existence of a nonconstant continuous function λ which allows us to show that the "cone" with the vertex $\psi(z_0)$ and the "base" $\psi\big(K(\bar{x},\varrho)\big)$ (having the non-void interior in the topology of R^n) is contained in $Q(H)$. Thus $Q(H)$, and hence H, belongs to $\mathcal{A}_n$, which was to be proved.

It should be observed that our theorem does not contain the theorem from [4] as a particular case, as we require stronger regularity assumptions.

R E F E R E N C E S

[1] R. G e r: Some new conditions of continuty of convex functions. Mathematica (Cluj) 12 (1970) 271-277.

[2] R. G e r, M. K u c z m a: On the boundedness and continuity of convex functions and additive functions. Aequationes Math. 4 (1970) 157-162.

[3] M. K u c z m a: Convex functions, Summer School on Functional Equations and Inequalities (La Mendola, August 1970). Proceedings (1971) 197-213

[4] M. K u c z m a: On some set classes occuring in the theory of convex functions. Comment Math. Prace Mat. (in print)

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