Title: On a composite functional equation

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ON A COMPOSITE FUNCTIONAL EQUATION

Abstract. We determine all continuous functions $f : (0, \infty) \longrightarrow (0, \infty)$ satisfying the functional equation

$$f(xG(f(x))) = f(x)G(f(x))$$

where G is continuous and strictly increasing function such that $1 \in G((0, \infty))$.

1. Introduction

We deal with continuous solution of the composite functional equation

$$f(xG(f(x))) = f(x)G(f(x)) \tag{1}$$

where $f : (0, \infty) \longrightarrow (0, \infty)$ is an unknown function. In the case when a given G is a power function this functional equation was considered in [2].

In the present paper, assuming that $G : (0, \infty) \longrightarrow (0, \infty)$ is continuous, strictly increasing and such that $1 \in G(0, \infty)$, we determine all continuous and strictly increasing solutions of this functional equation.

Note that (cf. also [2]) if $f : (0, \infty) \longrightarrow (0, \infty)$ is a bijective solution of the above functional equation, then the function $\phi := f^{-1}$ satisfies the following (non–composite!) linear homogenous iterative functional equation

$$\phi(xG(x)) = G(x)\phi(x).$$

Since the theory such equations is well-known (cf. M. Kuczma [3] and M. Kuczma, B. Choczewski, R. Ger [4]), we are mainly interested in noninvertible solution of the considered equation.

Let us mention that in the case when $G(u) = u^2$ equation (1) appears in a division model of population (cf. [1]).

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2. Main result

Our aim is to prove the following

THEOREM. *Suppose that* $G : (0,\infty) \longrightarrow (0,\infty)$ *is continuous, strictly increasing, and there exists a* $\gamma > 0$ *such that* $G(\gamma) = 1$. *A continuous function* $f : (0,\infty) \longrightarrow (0,\infty)$ *satisfies the functional equation*

$$f\left(xG(f(x))\right) = f(x)G\left(f(x)\right), \qquad x > 0, \tag{2}$$

if, and only if, there exist a, $b \in [0,+\infty]$, $a \le b$, *and* $a \ne b$ *if* $a = 0$ *or* $b = \infty$, *such that*

$$f(x) = \begin{cases} \frac{\gamma}{a}x & 0 < x \le a \\ \gamma & a < x < b \\ \frac{\gamma}{b}x & x \ge b. \end{cases} \tag{3}$$

Proof. Define the functions M, $D : (0,\infty) \longrightarrow (0,\infty)$ by

$$M(x) := xG\left(f(x)\right), \quad D(x) := \frac{f(x)}{x}, \qquad x > 0. \tag{4}$$

We can write equation (1) in the form

$$D\left(M(x)\right) = D(x), \quad x > 0. \tag{5}$$

If $M(x_1) = M(x_2)$ for some x_1, $x_2 > 0$, then, by (5), we get $D(x_1) = D(x_2)$, and, consequently, $D(x_1)M(x_1) = D(x_2)M(x_2)$. In view of the definitions of M and D it means that $f(x_1)G(f(x_1)) = f(x_2)G(f(x_2))$. Since the function $xG(x)$ is strictly increasing, it follows that $f(x_1) = f(x_2)$. Now the equality $D(x_1) = D(x_2)$ implies that $x_1 = x_2$. Thus M is one-to-one, and, by the continuity of G, M is strictly monotonic.

Suppose first that M is strictly increasing and put

$$Fix(M) := \{x > 0 : M(x) = x\}.$$

It is easy to see that

$$Fix(M) = \{x > 0 : f(x) = \gamma\}.$$

We shall prove that $Fix(M)$ is a nonempty, closed subinterval of $(0,\infty)$.

For an indirect argument first suppose that $Fix(M) = \emptyset$. The continuity of M implies that either $M(x) < x$, $(x > 0)$, or $M(x) > x$, $(x > 0)$. Hence, by definition (4) of M, either

$$G\left(f(x)\right) < 1, \quad x > 0,$$

or

$$G\left(f(x)\right) > 1, \quad x > 0.$$

Since $G(\gamma) = 1$, by the monotonicity of G, we infer that either

$$f(x) < \gamma, \quad x > 0; \tag{6}$$

or

$$f(x) > \gamma, \qquad x > 0. \tag{7}$$

On the other hand, the continuity of M and D, the monotonicity of M, and equation (5), imply that

$$D((0,\infty)) = D([M(1),1]).$$

Hence, setting

$$c := \inf D([M(1),1]), \qquad C := \sup D([M(1),1]),$$

we obtain the inequality $0 < c \leq D(x) \leq C < \infty$ for all $x > 0$ i.e.

$$0 < cx \leqslant f(x) \leqslant Cx < \infty, \qquad x > 0,$$

which contradicts (6), as well as (7). This proves that $Fix(M) \neq \emptyset$.

To show that $Fix(M)$ is an interval, for an indirect proof, suppose that there exists an interval $[c,d]$, $c < d$, such that $c, d \in Fix(M)$, and $(c,d) \cap Fix(M) = \emptyset$. Consequently, either $M(x) < x$ for all $x \in (c,d)$, or $M(x) > x$ for all $x \in (c,d)$. In the first case we would have

$$\lim_{n\to\infty} M^n(x) = c, \qquad x \in [c,d).$$

From equation (5), by induction, for every integer n, we get

$$D(x) = D(M^n(x)), \qquad x > 0.$$

The continuity of D implies

$$D(x) = \lim_{n\to\infty} D(M^n(x)) = D(c), \qquad x \in [c,d).$$

Hence, again by the continuity of D, we get $D(c) = D(d)$, i.e. that

$$f(c)d = f(d)c.$$

On the other hand we have $M(c) = c$ and $M(d) = d$, which means that

$$G(f(c)) = 1, \qquad G(f(d)) = 1.$$

Since G is one-to-one, it follows that $f(c) = f(d)$. Consequently $c = d$. This contradiction proves that $Fix(M)$ is an interval. If $M(x) > x$ we argue in the same way.

Put

$$a := \inf Fix(M), \qquad b := \sup Fix(M).$$

According to what we have proved,

$$0 \leqslant a < +\infty, \qquad 0 < b \leqslant +\infty.$$

Since M is continuous we have

$$Fix(M) = [a,b] \cap (0,\infty).$$

Hence,

$$f(x) = \gamma, \qquad x \in [a,b] \cap (0,+\infty). \tag{8}$$

If $b < +\infty$ then we have either $M(x) < x$ for all $x > b$, or $M(x) > x$ for all $x > b$. Suppose that $M(x) < x$ for all $x > b$. Then, for a fixed $x > b$,

$$\lim_{n\to\infty} M^n(x) = b.$$

Hence, by (5) and the continuity of D,

$$D(x) = \lim_{n\to\infty} D\left(M^n(x)\right) = D(b), \qquad x > b.$$

Suppose that $M(x) > x$ for all $x > b$. Then, for a fixed $x > b$,

$$\lim_{n\to\infty} M^{-n}(x) = b$$

and, for the same reason,

$$D(x) = \lim_{n\to\infty} D\left(M^{-n}(x)\right) = D(b), \qquad x > b.$$

Now the definition of D and the relation $b \in Fix(M)$ imply

$$f(x) = b^{-1}f(b)x = b^{-1}(\gamma)\,x, \qquad x > b.$$

If $a > 0$, we show in the same way that

$$f(x) = a^{-1}f(a)x = a^{-1}(\gamma)\,x, \qquad 0 < x < a.$$

Thus, if $0 < a \leqslant b < +\infty$ then we arrive at formula (3) for f. If $a = 0$ and $b = \infty$ obviously $f(x) = \gamma$, $x \in (0,\infty)$, in accordance with (3), too.

On the other hand, it is easy to verify that the functions given by this formula satisfy equation (1).

Now suppose that M is strictly decreasing. Then, by the definition of M, the function $G \circ f$ is also strictly decreasing. Because G is strictly increasing, so f is strictly decreasing. This is a contradiction because the function $f \circ M$, the left–hand side of equation (1), is strictly increasing, and the function $f \cdot (G \circ f)$, the right–hand side of equation (1), is strictly decreasing.

This completes the proof.

Remark 1. The assumption that the function G is strictly increasing is essential. It is a consequence of point 2^o and 3^o of Theorem 1 in [2] where $G(u) = u^{-2}$ or $G(u) = u^{-1}$, $u > 0$.

In the case when $G(u) = u^{-2}$, besides functions given by (3), for every continuous function $f_1 : [1,\infty) \to [1,\infty)$ such that $f_1(1) = 1$, and

$$x \to \frac{f_1(x)}{x} \text{ is increasing on } [1,\infty),$$

there exists a unique continuous solution f of equation (1) such that $f(x) = f_1(x)$ for all $x \geq 1$; moreover, the function f is an increasing homeomorphic mapping of $(0, \infty)$ onto itself.

In the case when $G(u) = u^{-1}$, a continuous $f : (0, \infty) \to (0, \infty)$ satisfies (1) if, and only if, there are $a,\ b \in [0, \infty)$, $a \leq b$, and $a \neq b$ if $a = 0$ or $b = \infty$; and continuous functions $f_a : (0, a] \to (0, \infty)$, $f_b : [b, \infty) \to (0, \infty)$ satisfying the conditions

$$\frac{x}{b} \leq f_a(x) \leq \frac{x}{a}, \quad x \in (0, a]; \quad \frac{x}{b} \leq f_b(x) \leq \frac{x}{a}, \quad x \in (b, \infty];$$
$$\lim_{x \to a^-} f_a(x) = 1 = \lim_{x \to b^+} f_b(x)$$

such that

$$f(x) = \begin{cases} f_a(x) & 0 \leq x < a \\ 1 & a \leq x \leq b \\ f_b(x) & x > b. \end{cases}$$

Thus, these two cases show that if the function G is not increasing, besides (3), some other type of solutions may appear.

REMARK 2. If G is constant, say $G = c$, $c \in (0, \infty)$, then equation (2) becomes $f(cx) = cf(x)$, $x \in (0, \infty)$, and the continuous solution of this equation depends on an arbitrary function (cf. M. Kuczma [3]). Thus the strict monotonicity of G in the theorem is indispensable.

REMARK 3. The assumption that $1 \in G((0, \infty))$ is also essential. It is easily seen from equation (2) that if $1 \notin G(0, \infty)$ then there is not a constant solution.

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