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ON A LOCAL STABILITY OF THE JENSEN FUNCTIONAL EQUATION

The problem of stability of Cauchy's functional equation on a restrict domain D in R has a positive answer [5]. In this paper we shall show that Cauchy's functional equation and Jensen's functional equation on a restrict domain D being some subset of R^N are stable. Moreover, we shall use a such type result to give a positive answer to a problem of K. Nikodem ([3], [4]).

Let $(X, \|\cdot\|)$ be a real Banach space and let D be a subset of X. We say that a function $f: D \longrightarrow X$ is ε-additive ($\varepsilon \geqslant 0$ is fixed) in D iff

(1) $$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon$$

for all $x, y \in D$ such that $x+y \in D$. Similarly, we say that $f: D \longrightarrow X$ is ε-Jensen function iff

(2) $$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leqslant \varepsilon$$

for all $x, y \in D$ such that $\frac{x+y}{2} \in D$. If $D = R^N$ and (1) holds true with $\varepsilon = 0$ we say that f is additive function and every function $f: D \longrightarrow X$ satisfying (2) with $\varepsilon = 0$ is called Jensen function.

F. Skof in [5] has proved that if $f: [0,a) \longrightarrow X$, $a > 0$, is ε-additive then there exists an additive function $F: R \longrightarrow X$ such that $\|f(x) - F(x)\| \leqslant 3\varepsilon$ for each $x \in [0,a)$.

1. ε-additive functions

We start with a theorem extending the result of F. Skof mentioned above.

Theorem 1. Let $f:[0,a)^N \longrightarrow X$, $a>0$, N - positive integer, be an ε-additive function in $[0,a)^N$. Then there exists an additive function $F: \mathbb{R}^N \longrightarrow X$ such that $\|f(x) - F(x)\| \leq (4N-1)\varepsilon$ for every $x \in [0,a)^N$.

Proof. We define the functions $f_i:[0,a) \longrightarrow X$, $i = 1,\dots,N$, by the following formulas:

$$f_i(x_i) := f(0,\dots,0,x_i,0,\dots,0).$$

The functions f_i, $i=1,\dots,N$, are ε-additive in $[0,a)$, thus the result of F. Skof mentioned above guarantees the existence of additive functions $F_i:\mathbb{R} \longrightarrow X$ for which the following inequalities

$$(3) \qquad \|F_i(x_i) - f_i(x_i)\| \leq 3\varepsilon, \quad x_i \in [0,a), \quad i = 1,\dots,N,$$

hold true.

Writting $x \in \mathbb{R}^N$ in the form $(x_1,\dots,x_N)$ we see that $F: \mathbb{R}^N \longrightarrow X$ given by the formula

$$(4) \qquad F(x) := \sum_{i=1}^{N} F_i(x_i)$$

is an additive function. Moreover, for each $x \in [0,a)^N$, by (3) and (1) we get

$$\|F(x) - f(x)\| \leq \sum_{i=1}^{N} \|F_i(x_i) - f_i(x_i)\| + \Big\| \sum_{i=1}^{N} f_i(x_i) - f(x)\Big\| \leq$$

$$\leq 3N\varepsilon + \Big\|\sum_{i=1}^{N-1} f_i(x_i) - f(x_1,\dots,x_{N-1},0)\Big\| + \|f(x_1,\dots,x_{N-1},0) +$$

$$+ f(0,\dots,0,x_N) - f(x)\| \leq 3N\varepsilon + \varepsilon + \Big\|\sum_{i=1}^{N-1} f_i(x_i) - f(x_1,\dots,x_{N-1},0)\Big\| \leq$$

$$\leq \dots \leq 3N\varepsilon + (N-1)\varepsilon = (4N-1)\varepsilon.$$

This completes the proof.

L e m m a 1. If a function $f:(-a,a) \longrightarrow X$ is ε-additive in $(-a,a)$ then there exists an additive function $F: R \longrightarrow X$ such that

$$\|F(x) - f(x)\| \leq 4\varepsilon \text{ for all } x \in (-a,a). \tag{5}$$

P r o o f . Putting

$$f_1(x) := \frac{1}{2}\left[f(x) + f(-x)\right], \quad x \in (-a,a)$$

and

$$f_2(x) := \frac{1}{2}\left[f(x) - f(-x)\right], \quad x \in (-a,a)$$

we note that

$$\|f_1(x)\| \leq \varepsilon, \text{ for every } x \in (-a,a), \tag{6}$$

and

$\|f_2(x+y) - f_2(x) - f_2(y)\| \leq \varepsilon$, for all $x,y \in (-a,a)$ such that $x+y \in (-a,a)$. On account of a result of F. Skof there exists an additive function $F: R \longrightarrow X$ such that $\|F(x)-f_2(x)\| \leq 3\varepsilon$ for each $x \in [0,a)$. But f_2 and F are odd and therefore

$$\|F(x) - f_2(x)\| \leq 3\varepsilon, \text{ for all } x \in (-a,a).$$

Hence, and by (6) we get condition (5).

L e m m a 2. If $f:(-a,a)^N \longrightarrow X$ is an ε-additive in $(-a,a)^N$ then there exists an additive function $F: R^N \longrightarrow X$ such that

$$\|F(x) - f(x)\| \leq (5N-1)\varepsilon, \quad \text{for any } x \in (-a,a)^N.$$

P r o o f . It is quite similar to the proof of Theorem 1. Here, we use Lemma 1 instead of a result of F.Skof.

T h e o r e m 2. Let $D \subset R^N$ be a bounded set containing zero in its interior. If, moreover,

$$\frac{1}{2} D \subset D \tag{i}$$

and $f: D \longrightarrow X$ is ε-additive in D then there exist additive function $F: R^N \longrightarrow X$ and constant $K > 0$ such that

$$\|F(x) - f(x)\| \leq K, \quad \text{for all} \quad x \in D.$$

More precisely, if non-negative integer p and positive a are such that the conditions

(ii) $$(-a,a)^N \subset D;$$

(iii) $$2^p(-a,a)^N \supset D$$

hold, then $\|F(x) - f(x)\| \leq (2^p 5N-1)\varepsilon$, for all $x \in D$.

P r o o f . Assume that a and p fulfil the conditions (ii) and (iii). On account of Lemma 2, we get the existence of an additive function $F: R^N \longrightarrow X$ for which the inequality

(7) $$\|F(x) - f(x)\| \leq (5N-1)\varepsilon, \quad x \in (-a,a)^N,$$

holds true. Taking an arbitrary $x \in D$ we observe, by (i), that $\frac{1}{2^k} x \in D$ for any $k \in \{1,2,\dots,p\}$, and condition (iii) implies that $\frac{1}{2^p} x \in (-a,a)^N$. It follows from (1) that for every $x \in D$ and each $k \in \{1,2,\dots,p\}$

$$\left\| 2^{k-1} f\left(\frac{x}{2^{k-1}}\right) - 2^k f\left(\frac{x}{2^k}\right)\right\| \leq 2^{k-1}\varepsilon$$

and therefore

(8) $$\left\| f(x) - 2^p f\left(\frac{x}{2^p}\right)\right\| \leq (2^p-1)\varepsilon.$$

Now, by (7), (8) and the additivity of F we get

$$\|F(x) - f(x)\| \leq \left\| 2^p F\left(\frac{x}{2^p}\right) - 2^p f\left(\frac{x}{2^p}\right)\right\| + \left\| 2^p f\left(\frac{x}{2^p}\right) - f(x)\right\| \leq$$

$$\leq 2^p(5N-1)\varepsilon + (2^p-1)\varepsilon = (2^p 5N-1)\varepsilon,$$

for any $x \in D$, which ends the proof.

2. ε-Jensen functions

L e m m a 3. If $g: (-a,a)^N \longrightarrow X$, $a > 0$, is an ε-Jensen function then there exists a Jensen function $G: R^N \longrightarrow X$ such that

(9) $\|G(x) - g(x)\| \leq (25N-4)\varepsilon$, for all $x \in (-a,a)^N$.

P r o o f . The function $f_1:(-a,a)^N \longrightarrow X$ defining by the formula $f_1(x) := g(x) - g(0)$ is ε-Jensen function, too, and moreover, $f_1(0) = 0$. We define a function $f:(-a,a)^N \longrightarrow X$ in the following manner: for any positive integer n and $x \in A_n := \left(-\frac{a}{2^{n-1}}, \frac{a}{2^{n-1}}\right)^N \setminus \left(-\frac{a}{2^n}, \frac{a}{2^n}\right)^N$ we put $f(x) :=$ $:= \frac{1}{2^{n-1}} f_1(2^{n-1}x)$.

According to (2) and $f_1(0) = 0$, for every $y \in (-a,a)^N$ and each positive integer k we have

$$\left\| f_1(y) - 2^k f_1\left(\frac{y}{2^k}\right)\right\| \leq (2^k-1)\varepsilon.$$

Hence, and by the definition of f, if $x \in A_n$ then

(10) $$\|f(x) - f_1(x)\| \leq \varepsilon.$$

Moreover,

$$f(x) = 2f\left(\frac{x}{2}\right) \quad \text{for any} \quad x \in (-a,a)^N.$$

The last equality together with (10) and (2) (for the function f_1) imply that

$$\|f(x+y) - f(x) - f(y)\| = \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq$$

$$\leq 2\left\|f\left(\frac{x+y}{2}\right) - f_1\left(\frac{x+y}{2}\right)\right\| + \|f(x) - f_1(x)\| + \|f(y) - f_1(y)\| +$$

$$+ \left\|2f_1\left(\frac{x+y}{2}\right) - f_1(x) - f_1(y)\right\| \leq 5\varepsilon$$

for all $x,y \in (-a,a)^N$ such that $x+y \in (-a,a)^N$, which means that f is 5ε-additive in $(-a,a)^N$. By Lemma 2 we infer that there exists an additive function $F: R^N \longrightarrow X$ such that

(11) $\|F(x) - f(x)\| \leq (5N-1)5\varepsilon$, for all $x \in (-a,a)^N$.

Putting

$$G(x) := F(x) + g(0), \quad x \in R^N,$$

we see that G is a Jensen function and (see (10) and (11))

$$\|G(x) - g(x)\| = \|F(x) - f_1(x)\| \leq 25N\varepsilon - 5\varepsilon + \varepsilon = (25N-4)\varepsilon$$

for every $x \in (-a,a)^N$.

T h e o r e m 3. Let $D_1 \subset R^N$ be a bounded set with non-empty interior. If there exists an element $x_0 \in \operatorname{int} D_1$ such that the set $D := D_1 - x_0$ satisfies condition (i) then for every ε-Jensen function $g_1 : D_1 \longrightarrow X$ there exist Jensen function $G_1 : R^N \longrightarrow X$ and constant K (depending on g_1) such that

$$\|G_1(x) - g_1(x)\| \leq K \quad \text{for any} \quad x \in D_1.$$

More precisely, if non-negative integer p and positive a fulfil conditions (ii) and (iii) then

$$\|G_1(x) - g_1(x)\| \leq [2^p(25N-3)-1]\varepsilon \quad \text{for each} \quad x \in D_1.$$

P r o o f. Assume that a, p and D fulfil conditions (i), (ii) and (iii). The function

$$g(x) := g_1(x+x_0), \quad x \in D,$$

is an ε-Jensen function in D. Similarly as in the proof of Lemma 3 we define the functions f_1 and f, namely

$$f_1(x) := g(x) - g(0), \quad x \in D,$$

$$f(x) := \frac{1}{2^{n-1}} f_1(2^{n-1}x), \quad x \in A_n, \quad n = 1,2,\ldots .$$

We note that

$$(12) \qquad \|f_1(y) - 2^p f_1\left(\frac{y}{2^p}\right)\| \leq (2^p-1)\varepsilon, \quad \text{for each} \quad y \in D,$$

and (see (10))

$$(13) \qquad \|f(x) - f_1(x)\| \leq \varepsilon, \quad \text{for any} \quad x \in (-a,a)^N.$$

Let $F: R^N \longrightarrow X$ be an additive function such that (cf. (11))

$$\|F(x) - f(x)\| \leq (5N-1)5\varepsilon \quad \text{for all} \quad x \in (-a,a)^N.$$

Taking any $x \in D$, by (12) and (13), we observe that

(14) $$\|F(x) - f_1(x)\| \leq 2^p\left\|F\left(\frac{x}{2^p}\right) - f_1\left(\frac{x}{2^p}\right)\right\| + \left\|2^p f_1\left(\frac{x}{2^p}\right) - f_1(x)\right\| \leq$$

$$\leq 2^p\left(\left\|F\left(\frac{x}{2^p}\right) - f\left(\frac{x}{2^p}\right)\right\| + \left\|f\left(\frac{x}{2^p}\right) - f_1\left(\frac{x}{2^p}\right)\right\|\right) + \left\|2^p f_1\left(\frac{x}{2^p}\right) - f_1(x)\right\| \leq$$

$$\leq 2^p(25N-5+1)\varepsilon + (2^p-1)\varepsilon = [2^p(25N-3)-1]\varepsilon .$$

Now, we put

$$G_1(x) := F(x-x_0) + g(0), \quad x \in R^N.$$

Hence, by definitions of g and f_1 and (14) we get

$$\|G_1(x) - g_1(x)\| = \|F(x-x_0) + g(0) - g(x-x_0)\| =$$

$$= \|F(x-x_0) - f_1(x-x_0)\| < [2^p(25N-3)-1]\varepsilon ,$$

for any $x \in D_1$. Thus the proof is complete.

3. Application

A real function f defined on an open and convex subset D of R^N is said to be J-convex (convex in the sense of Jensen) if inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

holds for all $x,y \in D$. If the inequality sign "$\leq$" is replaced by "$\geq$" in the above inequality we say that f is a J-concave function. In the paper [1] R. Ger and M. Kuczma introduced the following class $\mathcal{A}$ of sets. A set $T \subset R^N$ belongs to the class $\mathcal{A}$ iff every J-convex function defined on a convex open domain $D \supset T$ bounded above on T is continuous in D. Some interesting results concerning of this class of sets may be found in [2].

Theorem 4. Let $D\subset R^N$ be an open convex set and let $T\subset D$ be a fixed set belonging to the class $\mathcal{A}$. If $f:D\to R$ is J-convex and $g:D\to R$ is J-concave and, moreover,

$$f(x)\leqslant g(x) \quad \text{for all} \quad x\in T$$

then there exist additive function $H: R^N\to R$, convex function $f_1:D\to R$ and concave function $g_1:D\to R$ such that $f(x) = H(x) + f_1(x)$ and $g(x) = H(x) + g_1(x)$ for every $x\in D$.

Proof. Putting

$$\varphi(x) := f(x) - g(x), \quad x\in D$$

we note that φ is J-convex function bounded above on T. Thus φ is continuous in D. Let D_1 be an open convex and bounded subset of D for which there exists a constant $M>0$ such that

$$|\varphi(x)|\leqslant M \quad \text{for any} \quad x\in D_1. \tag{15}$$

From the definitions of φ, J-concavity of g, J-convexity of f and (15) it follows that

$$0\leqslant 2g\left(\frac{x+y}{2}\right) - g(x) - g(y) =$$

$$= 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - 2\varphi\left(\frac{x+y}{2}\right) + \varphi(x) + \varphi(y)\leqslant 4M,$$

for all $x,y\in D_1$. In particular,

$$\left|2g\left(\frac{x+y}{2}\right) - g(x) - g(y)\right|\leqslant 4M \quad \text{for all} \quad x,y\in D_1.$$

On account of Theorem 3 there exist Jensen function $G: R^N\to R$ and constant $K>0$ such that

$$|G(x) - g(x)|\leqslant K \quad \text{for every} \quad x\in D_1. \tag{16}$$

Now, we define functions H, g_1 and f_1 by the formulas

$$H(x) := G(x) - G(0), \quad x\in R^N,$$

$$g_1(x) := g(x) - H(x), \quad x \in D,$$

$$f_1(x) := \varphi(x) + g_1(x), \quad x \in D.$$

Of course, H is an additive function. The function g_1 is J-concave bounded below on D_1 (cf. (16)) and therefore it is concave function on D. The function f_1 is convex, because it is continuous and $f_1(x) = \varphi(x) + g(x) - H(x) = f(x) - H(x)$, which imply that f_1 is J-convex. Moreover, it is easily seen that $f(x) = H(x) + f_1(x)$ and $g(x) = H(x) + g_1(x)$ for any $x \in D$.

Analogous result was obtained by K. Nikodem [4] and C.T. Ng [6] but the assumptions in Theorem 4 are slightly weaker and the method of the proof is completely different from that presented in both papers.

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