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1. Introduction

The study of random equations was initiated by the Prague school of probabilists in the Fifties. An account of results on this topic can be found in the books by Bharucha-Reid [1], Soong [15], Tsokos and Padgett [16]. In this paper we investigate the generalized differential equation $\dot{x} \in \varphi(\omega,t,x)$, where ω is a random parameter and φ is a given set-valued mapping. Deterministic equations of this type were studied e.g. by Filippov [8] and Hermes [9]. First results on random differential equations with the multivalued right-hand side were obtained by Castaing [3], [4] (see also [5]). Phan Van Chuong [13] studied generalized random integral equations.

In this paper we propose an approach to generalized random equations which is based on the measurable selection theorem. We assume that for each value of a random parameter the equation has a solution and prove the existence of a random solution. Similar approach was already used to various problems in stochastic analysis (see e.g. Bocsan [2], Engl [6], [7], Nowak [11], [12]).

2. Notation and definitions

Throughout the paper $(\Omega, \mathcal{U}, P)$ is a complete probability space, and T a closed bounded interval with the beginning at the point 0. For a metric space X, $\mathcal{B}_X$ denotes the Borel σ-field on X, and $\mathcal{F}(X)$ the family of all closed and non-

-empty subsets of X. By $\mathcal{U} \times \mathcal{B}_X$ we mean the product σ-field on $\Omega \times X$.

We shall consider the generalized random differential equation

(1) $$\dot{x}(t) \in \varphi(\omega, t, x(t)), \quad t \in T$$

with the initial condition

(2) $$x(0) \in \psi(\omega),$$

where $\varphi: \Omega \times T \times R^n \longrightarrow \mathcal{F}(R^n)$ and $\psi: \Omega \to \mathcal{F}(R^n)$ are given set-valued mappings.

A function $\xi: \Omega \times T \longrightarrow R^n$ is called a random solution of (1)-(2) if it is measurable in ω, continuously differentiable in t, and for all $\omega \in \Omega$,

(3) $$\frac{d}{dt}\xi(\omega, t) \in \varphi(\omega, t, \xi(\omega, t)), \quad t \in T$$

(4) $$\xi(\omega, 0) \in \psi(\omega).$$

Let $\Phi: \Omega \to \mathcal{F}(X)$ be a set-valued map. By the graph of Φ we mean

$$\mathrm{gr}\ \Phi = \{(\omega, x) \in \Omega \times X : x \in \Phi(\omega)\}.$$

We call Φ measurable if for all open $G \subset X$,

$$\{\omega \in \Omega : \Phi(\omega) \cap G \neq \emptyset\} \in \mathcal{U}.$$

A function $\eta: \Omega \longrightarrow X$ is a measurable selection of Φ if it is measurable and for all $\omega \in \Omega$, $\eta(\omega) \in \Phi(\omega)$.

Denote by $C(T)$ the space of all continuous functions $x: T \longrightarrow R^n$ endowed with the norm

$$\|x\| = \sup_{t \in T} |x(t)|,$$

where $|\cdot|$ is a norm in R^n. By $C^1(T)$ we mean the family of all continuously differentiable functions $x: T \longrightarrow R^n$ with the norm

$$\|x\|_1 = \|x\| + \|\dot{x}\|.$$

$C(T)$ and $C^1(T)$ are separable Banach spaces.

The generalized Hausdorff metric D on $\mathcal{F}(R^n)$ is defined by

$$D(A,B) = \max\left\{\sup_{u\in A} d(u,B),\ \sup_{u\in B} d(u,A)\right\},$$

where $A,B \in \mathcal{F}(R^n)$, and

$$d(u,A) = \inf_{v\in A} |u-v|.$$

We shall use the following inequality

(5) $|d(u,A) - d(v,B)| \leqslant |u-v| + D(A,B), \quad u,v \in R^n,\ A,B \in \mathcal{F}(R^n).$

3. Existence of random solutions

We assume that for each $\omega \in \Omega$ the problem (1)-(2) has a solution and we give sufficient conditions for the existence of a solution depending measurably on ω. The following lemma will be useful:

Lemma ([12], Lemma 3). Let $(X, \|\cdot\|_X)$ be a normed linear space in $C(T)$ which is stronger than $C(T)$, i.e. there is a constant $K > 0$ such that $\|x\| \leqslant K\|x\|_X$ for all $x \in X$. If $\eta : \Omega \to X$ is measurable, then the function $\xi : \Omega \times T \to R^n$ defined as $\xi(\omega,t) = \eta(\omega)(t)$ is measurable in ω.

Now we can state our main result.

Theorem. Let $\varphi : \Omega \times T \times R^n \to \mathcal{F}(R^n)$ and $\psi : \Omega \to \mathcal{F}(R^n)$ satisfy the following conditions:

(i) for each $\omega \in \Omega$, $\varphi(\omega,\cdot)$ is continuous in the generalized Hausdorff metric,

(ii) for each $(t,u) \in T \times R^n$, $\varphi(\cdot,t,u)$ is measurable,

(iii) ψ is measurable.

If for each $\omega \in \Omega$, the problem (1)-(2) has a continuously differentiable solution, then it has a random solution.

P r o o f . Define a new set-valued mapping Φ on Ω by $\Phi(\omega)=\{x\in C^1(T):\dot{x}(t)\in\varphi(\omega,t,x(t))$ for all $t\in T$, and $x(0)\in\psi(\omega)\}$.

Under our assumptions Φ has non-empty values. If $\eta:\Omega\to C^1(T)$ is a measurable selection of Φ, then $\xi(\omega,t)=\eta(\omega)(t)$ is a random solution. Indeed, ξ is measurable in ω (by Lemma), continuously differentiable in t, and satisfies (3)-(4).

We shall prove that the graph of Φ is $\mathcal{U}\times\mathcal{B}_{C^1(T)}$-measurable. Let the function $g:\Omega\times T\times C^1(T)\to R$ be defined by

$$g(\omega,t,x)=d(\dot{x}(t),\varphi(\omega,t,x(t)))+d(x(0),\psi(\omega)).$$

Since the set-valued mappings φ and ψ are measurable in ω, for each $(t,x)\in T\times C^1(T)$, $g(\cdot,t,x)$ is measurable (see e.g. [10], Theorem 3.3). By the inequality (5),

$$|g(\omega,t,x)-g(\omega,s,y)|\leq$$

$$\leq|\dot{x}(t)-\dot{y}(s)|+D(\varphi(\omega,t,x(t)),\varphi(\omega,s,y(s)))+|x(0)-y(0)|$$

for all $\omega\in\Omega$, $s,t\in T$ and $x,y\in C^1(T)$. It implies the continuity of $g(\omega,\cdot)$. Being measurable in ω and continuous in (t,x), g is jointly measurable.

Define the function $h:\Omega\times C^1(T)\to R$ as

$$h(\omega,x)=\sup_{t\in T}g(\omega,t,x).$$

We have

$$\mathrm{gr}\,\Phi=\{(\omega,x)\in\Omega\times C^1(T):h(\omega,x)=0\}.$$

Let E be a dense countable subset of T. Since g is continuous in t,

$$h(\omega,x)=\sup_{t\in E}g(\omega,t,x).$$

Thus h is measurable and, consequently, $gr\,\Phi \in \mathcal{U}\times\mathcal{B}_{C^1(T)}$. Hence, Φ admits a measurable selection (see [14], Theorem 3). This completes the proof.

Remarks: 1. For the probability space $(\Omega, \mathcal{U}, P)$ which is not necessarily complete, we can obtain a slight modification of our Theorem with a random solution ξ satisfying almost surely the conditions (3)-(4). The proof follows in the same way as the previous one, but instead of Theorem 3 we apply Corollary 1 from [14], and obtain a selector ζ of Φ which is measurable with respect to the completion $\mathcal{U}_P$ of the σ-field $\mathcal{U}$. There exists a function $\eta:\Omega \to C^1(T)$ measurable with respect to $\mathcal{U}$ and such that $\eta(\omega) = \zeta(\omega)$ a.s. It is obvious that $\xi(\omega,t) = \eta(\omega)(t)$ is the desired solution. In this case it suffices if all assumptions on the problem (1)-(2) are satisfied a.s.

2. Theorem holds if we replace the closed bounded interval T by the half-line $[0,\infty)$. In this case we consider $C[0,\infty)$ and $C^1[0,\infty)$ with the topology induced by the family of semi-norms

$$p_n(x) = \sup_{0\le t\le n} |x(t)|, \quad x \in C[0,\infty), \quad n=1,2,\ldots$$

and

$$q_n(x) = p_n(x) + p_n(\dot{x}), \quad x \in C^1[0,\infty), \quad n=1,2,\ldots,$$

respectively. It is known that these are separable Frechét spaces. The same proof holds.

As an example we apply our theorem to the following generalized random equation:

$$\dot{x}(t) \in \varphi(\omega,x(t)), \quad t\in T, \tag{6}$$

$$x(0) = f(\omega), \tag{7}$$

where $\varphi:\Omega\times R^n \to \mathcal{F}(R^n)$ and $f:\Omega\to R^n$ are given.

C o r o l l a r y . Suppose the problem (6)-(7) satisfies the following conditions:

(i) φ is compact-valued; for each $u \in R^n$, $\varphi(\cdot,u)$ is measurable, for each $\omega \in \Omega$, the set

$$\varphi(\omega,R^n) = \bigcup_{u \in R^n} \varphi(\omega,u)$$

is bounded, and there is $k(\omega) < \infty$ such that

$$D(\varphi(\omega,u),\ \varphi(\omega,v)) \leqslant k(\omega)\ |u-v|,\quad u,v \in R^n,$$

(ii) f is measurable.

Then the equation (6)-(7) has a random solution.

P r o o f . By a result of Hermes ([9], Theorem 1), for each $\omega \in \Omega$ the problem (6)-(7) has a solution in $C^1(T)$. In order to complete the proof it suffices to apply Theorem.

R e m a r k 3. Castaing [3], [4] (see also [5]) studied generalized random differential equations of a special type. He proved the existence of a solution ξ which is measurable in ω, absolutely continuous in t, and $P \times \lambda$-almost everywhere satisfies (3). Here λ is the Lebesgue measure on T.

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