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Author: Henryk Gacki

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Henryk Gacki

ON THE EXISTENCE AND UNIQUENESS OF A SOLUTION
OF THE RANDOM INTEGRAL EQUATION WITH ADVANCING ARGUMENT

1. Introduction

In this paper we shall prove two theorems on the existence of a unique random solution of the following random integral equation with advancing argument:

$$(1) \quad x(t, \omega) = h(t, \omega) + \int_0^{t+\delta(t)} K(t, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau$$

in the class of all continuous and bounded functions defined on R^+ with values in $L_2(\Omega, A, P)$, where (Ω, A, P) denotes the probability space.

Nonrandom differential equations with advancing argument have been investigated by other authors (see [1], [2], [3]).

The problem of the existence of a solution for random integral equation of the Volterra type with advancing argument has been considered in the class of all mappings $x: R_x^+ \Omega \rightarrow R$ such that, for every $t, x(t, \cdot)$ is a random variable (see [5]).

The fundamental theorems of this paper generalize some results of Christ. P. Tsokos and W. J. Padgett [6], by making use of certain ideas of that paper.

2. Preliminaries

We start by introducing the following definitions:

Definition 1. The function $x:R^+ \times \Omega \rightarrow R$ is called a stochastic process if for each $t \in R^+$, $x(t, \cdot)$ is a random variable defined on Ω .

Definition 2. A stochastic process $x(t, \omega), t \in R^+$, is said to belong to the space $L_2 = L_2(\Omega, A, P)$ or to be a second order if for each $t \in R^+$, we have

$$E(|x(t, \cdot)|^2) = \int_{\Omega} x(t, \omega)^2 dP(\omega) < \infty$$

We introduce a norm in the following way:

$$\|x(t, \cdot)\|_{L_2} = (E|x(t, \cdot)|^2)^{\frac{1}{2}}.$$

Thus, for each $t \in R^+$, L_2 is a Banach space.

Definition 3. The function $x: \Omega \rightarrow R$ is called P-essentially bounded if there is a constant $\alpha > 0$ such that,

$$P\{\omega: |x(\omega)| > \alpha\} = 0.$$

The space of all such functions will be denoted by $L_{\infty} = L_{\infty}(\Omega, A, P)$, the norm in this space is given by the formula:

$$\|x\| = \inf_{\Omega_0 \subset \Omega, P(\Omega_0) = 0} \sup_{\omega \in \Omega \setminus \Omega_0} |x(\omega)|,$$

where $P(\Omega_0) = 0$.

Definition 4. By $C(R^+, L_2(\Omega, A, P))$ we denote the space of all continuous and bounded functions x defined on R^+ with values in $L_2(\Omega, A, P)$. The norm in this space is defined by the formula

$$\|x\|_C = \sup_{t \geq 0} \|x(t, \omega)\|_{L_2(\Omega, A, P)}.$$

Definition 5. We shall denote by $C_g(R^+, L_2)$ the space of all continuous functions x from R^+ into L_2 such that there exists a positive number z and a positive continuous function g defined on R^+ , satisfying the inequality

$$\left(\int_{\Omega} |x(t, \omega)|^2 dP(\omega) \right)^{\frac{1}{2}} \leq z g(t) \text{ for every } t \in R^+.$$

In C_g the norm will be defined as follows

$$\|x\|_{C_g} = \sup_{t \geq 0} \frac{\|x(t, \cdot)\|_{L_2}}{g(t)}.$$

Definition 6. Let $C_c(R^+, L_2)$ denote the space of all continuous functions x on R^+ into L_2 . On this space is defined a family of seminorms:

$$\|x\|_n = \sup_{0 \leq t \leq n} \left(\int_{\Omega} |x(t, \omega)|^2 dP(\omega) \right)^{\frac{1}{2}} \quad n=1, 2, \dots$$

The topology on C_c is defined by the following distance function

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}.$$

Having the topology defined above, we may treat $C_c(R^+, L_2)$ as a complete metric space. Hence, $C(R^+, L_2)$ is a subspace of $C_c(R^+, L_2)$.

Definition 7. The random process x is a random solution of the equation (1) if for every fixed $t \in R^+$ $x(t, \cdot) \in L_2$ and $x(t, \cdot)$ satisfies (1) a.s.

Now, let (B, D) be a pair of Banach spaces such that $B, D \in C_c(R^+, L_2)$ and let T be a linear operator from $C_c(R^+, L_2)$ into itself.

D e f i n i t i o n 8. The pair of spaces (B, D) is called admissible with respect to the operator $T: C_c(R^+, L_2) \rightarrow C_c(R^+, L_2)$ if $T(B) \subset D$.

3. Existence theorems

The main result of the paper are the following theorems:

T h e o r e m 1. Suppose the following assumptions are satisfied:

1° There exists a function $a: R^+ \rightarrow R^+$ and a constant $\Lambda, 0 < \Lambda < e^{-1}$ such that

$$|b(t+\delta(t)) - b(t)| < \Lambda, \text{ for every } t \in R^+, \text{ where } b(t) = \int_0^t a(\tau) d\tau.$$

2° The function $\delta(t)$ is nonnegative, continuous for $t \in R^+$.

3° For every $t, \tau \in R^+$ such that $0 < \tau \leq t + \delta(t) < \infty$ the stochastic kernel $K(t, \tau, \cdot)$ is a random variable satisfying the condition:

$$\| \| K(t, \tau, \cdot) \| \| \leq a(\tau) e^{-b(t)}.$$

4° The function $f: R^+ \times R \rightarrow R$ is continuous with respect to $t \in R^+$ uniformly in scalars $x \in R$, such that $f(t, 0) \in C_g$ and f satisfies the following Lipschitz condition:

$$|f(t, x) - f(t, y)| \leq \lambda g(t) |x - y|,$$

where $0 \leq \lambda < e^{1-e\Lambda}$ and

$$g(t) = e^{eb(t)}.$$

5° Let h be a continuous, bounded function from R^+ , into L_2 thus $h \in C(R^+, L_2)$.

6° Moreover, for any fixed p the inequality

$$\|h\|_C + L\|f_0\|_{C_g} \leq p(1 - \lambda e^{a\lambda - 1}), \quad L = e^{a\lambda - 1}, \quad f_0 = f(t, 0)$$

holds.

Then the equation (1) has a unique random solution, $x \in C$ such that

$$\|x\|_C \leq p$$

P r o o f . At first we shall prove that under the assumptions 1°, 3° the pair (C_g, C) is admissible with respect to the operator T defined as follows:

$$(Tx)(t, \omega) = \int_0^{t+\delta(t)} K(t, \tau, \omega)x(\tau, \omega) d\tau, \quad \text{where } x \in C_g.$$

In virtue of a generalized Holder's inequality (see [4], p.530) we have

$$(2) \quad \begin{aligned} \|(Tx)(t, \cdot)\|_{L_2} &= \left(\int_{\Omega} \left(\int_0^{t+\delta(t)} K(t, \tau, \omega)x(\tau, \omega) d\tau \right)^2 dP(\omega) \right)^{\frac{1}{2}} \\ &\leq \int_0^{t+\delta(t)} \|K(t, \tau, \cdot)\| \cdot \|x(\tau, \cdot)\|_{L_2} d\tau. \end{aligned}$$

This inequality and the assumptions 1° and 3° imply

$$(3) \quad \|(Tx)(t, \cdot)\|_{L_2} \leq \|x\|_{C_g} e^{-1} e^{[b(t+\delta(t)) - b(t)]} \leq \|x\|_{C_g} e^{a\lambda - 1}.$$

Hence, $\|(Tx)(t, \cdot)\|_{L_2}$ is bounded, so $Tx \in C$ for every $x \in C_g$. Therefore $T(C_g) \subset C$ and (C_g, C) is admissible with to T .

Chris. P. Tsokos and W.J. Padgett proved in [6], p.36 that under the assumption 4^o if $x \in C$ then $f(t, x(t, \omega)) \in C_g$ and the following inequality holds

$$(4) \quad \|f(t, x) - f(t, y)\|_{C_g} \leq \lambda \|x - y\|_C \quad \text{for every } x, y \in S,$$

where

$$S = \{x \in C: \|x\|_C \leq p\}.$$

Now, define the operator $U: S \rightarrow C$ by

$$(Ux)(t, \omega) = h(t, \omega) + \int_0^{t+\delta(t)} K(t, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau.$$

The operator U has the following properties:

(i) $U(S) \subset S$

(ii) U is a contraction on S .

We shall prove (i). Let $x \in S$. Applying assumptions 1^o, 3^o, 4^o and (4), analogously as in (2) we obtain

$$\|U(x(t, \cdot))\|_{L_2} \leq \int_0^{t+\delta(t)} \|K(t, \tau, \cdot)\| \cdot \|f(\tau, \cdot)\|_{L_2} d\tau + \|h(t, \cdot)\|_{L_2} \leq$$

$$\leq \int_0^{t+\delta(t)} a(\tau) e^{-b(t)} e^{eb(\tau)} d\tau \|f\|_{C_g} + \|h(t, \cdot)\|_{L_2} \leq$$

$$\leq \|h(t, \cdot)\|_{L_2} + e^{a\lambda-1} \|f\|_{C_g} \leq$$

$$\leq \|h(t, \cdot)\|_{L_2} + e^{a\lambda-1} \|f - f_0\|_{C_g} + e^{a\lambda-1} \|f_0\|_{C_g} \leq$$

$$\leq \|h(t, \cdot)\|_{L_2} + e^{a\lambda-1} \lambda \|x\|_C + \|f_0\|_{C_g} \cdot e^{a\lambda-1}.$$

This implies using 6^o

$$\|Ux\|_C \leq \|h\|_C + e^{e\Lambda-1} \lambda p + e^{e\Lambda-1} \|f_0\|_{C_g} \leq e^{e\Lambda-1} \lambda p + p(1 - e^{e\Lambda-1} \lambda) = p.$$

Hence, $Ux \in S$ for all $x \in S$, thus $U(S) \subset S$.

Now, we shall prove (ii). Let $x, y \in S$; applying the generalized Holder's inequality, assumptions 1^o, 2^o, 3^o and the inequality (3) we have

$$\begin{aligned} \|(Ux)(t, \cdot) - (Uy)(t, \cdot)\|_{L_2} &\leq \left(\int_{\Omega} \left(\int_0^{t+\delta(t)} K(t, \tau, \omega) |f(\tau, x) - f(\tau, y)| d\tau \right)^2 dP(\omega) \right)^{\frac{1}{2}} \\ &\leq \int_0^{t+\delta(t)} \| \|K(t, \tau, \cdot)\| \cdot \|f(\tau, x) - f(\tau, y)\|_{L_2} d\tau \leq \\ &\leq e^{e\Lambda-1} \|f(\tau, x) - f(\tau, y)\|_{C_g} \leq e^{e\Lambda-1} \lambda \|x - y\|_C. \end{aligned}$$

This inequality implies that:

$$\|Ux - Uy\|_C \leq k \|x - y\|_C, \quad \text{where } k = \lambda e^{e\Lambda-1},$$

so, by 4^o, U is a contraction operator on S .

Now, we may notice, that from the Banach fixed point theorem, there exists a unique fixed point of U in S , which is a unique random solution of the equation (1).

This fact completes the proof.

We shall close this paper with the analogous theorem dealing with the case when $g(t) = 1$, for all $t \in R^+$.

Theorem 2. Let us assume that the following conditions hold for the random equation (1):

1^o There exists a function $a: R^+ \rightarrow R^+$ and a constant Λ , $0 < \Lambda < e^{-1}$ such that

$$|b(t+\delta(t)) - b(t)| < \Lambda, \quad \text{for every } t \in R^+, \text{ where } b(t) = \int_0^t a(\tau) d\tau.$$

2° The function $\delta(t)$ is nonnegative, continuous for $t \in \mathbb{R}^+$.

3° For every $t, \tau \in \mathbb{R}^+$ such that $0 \leq \tau \leq t + \delta(t) \leq \infty$ the stochastic kernel $K(t, \tau, \cdot)$ is a random variable satisfying the condition

$$\| \| K(t, \tau, \cdot) \| \| \leq a(\tau) e^{[b(\tau) - b(t)]}$$

4° The function $f: \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous uniformly in x and satisfies the following condition

$$|f(t, x) - f(t, y)| \leq \lambda |x - y|, \text{ where } 0 < \lambda < e^{1 - e^\Lambda}.$$

5° Let h be a continuous and bounded function from \mathbb{R}^+ into L_2 such that for fixed p the inequality

$$\| h \|_C + e^{1 - e^\Lambda} \| f_0 \|_C \leq p(1 - e^{e^\Lambda - 1})$$

holds.

Then the equation (1) has a unique random solution x , belonging to the class C and such that $\| x \|_C \leq p$.

P r o o f . We need to show that the pair (C, C) is admissible with respect to the integral operator T , defined as follows

$$(Tx)(t, \omega) = \int_0^{t + \delta(t)} K(t, \tau, \omega) x(\tau, \omega) d\tau, \text{ for } x \in C.$$

Indeed, let $x \in C$. Then by the condition 1° and the generalized Holder's inequality we have:

$$\begin{aligned} \| Tx(t, \cdot) \|_{L_2} &= \int_0^{t + \delta(t)} \| \| K(t, \tau, \cdot) \| \| \cdot \| x(\tau, \cdot) \|_{L_2} d\tau \leq \\ &\leq \| x \|_C \int_0^{t + \delta(t)} a(\tau) e^{[b(\tau) - b(t)]} d\tau \leq e^{e^\Lambda - 1} \| x \|_C \end{aligned}$$

Using the last inequality we obtain

$$\|Tx\|_C \leq e^{e\Lambda-1} \|x\|_C.$$

This inequality implies that $Tx \in C$ for every $x \in C$. Therefore the pair (C, C) is admissible with respect to T .

The remainder of the proof follows in a fashion similar to the proof of Theorem 1 and is omitted.

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INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY, KATOWICE

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