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ON THE SOLUTIONS OF THE EQUATION $f(xf(y)^k+yf(x)^l)=f(x)f(y)$

The functional equation

$$f(xf(y)^k + yf(x)^l) = f(x)f(y), \tag{1}$$

where k and l are positive integers and the unknown function f maps $\mathbb{R}$ into itself, has appeared in connection with determining some subsemigroups of the group L_2^1 (cf. [2]). Putting $k = 0$ and $l = 1$ we get the Gołąb-Schinzel equation as a particular case of (1) which has been studied by many authors including N. Brillouët who in [1] has also dealt with continuous solutions of equation

$$f(xf(y) + yf(x)) = \alpha f(x)f(y).$$

Our results presented here generalize those from [4] and [1] (in the case $\alpha = 1$). They are also more general than it was announced by M. Sablik at the 21st Symposium on Functional Equations (cf. [3]).

Let X be a linear space over reals. We will look for the real-valued solutions of (1) defined on X. We have the following obvious lemma.

Lemma 1. If $f: X \to \mathbb{R}$ is a solution of (1), then $f(0) \in \{0,1\}$.

Let us prove a very fruitful lemma.

Lemma 2. If $f: X \to \mathbb{R}$ is a solution of (1) and $c \in (0,1)$ is such that $c^k + c^l = 1$, then $f(x) \neq c$ for all $x \in X$.

P r o o f . Suppose $f(z) = c$ for a $z \in X$. Put $x = y = z$ in (1). We derive that $c = c^2$ i.e. $c \in \{0,1\}$ which is a contradiction.

Observe that $f(X)$ is a multiplicative semigroup. Hence in particular $f(X)$ contains all powers of its elements. Thus $f(X)$ cannot contain $\sqrt[n]{c}$ for any $n \in \mathbb{N}$ and $c \in (0,1)$ such that $c^k + c^l = 1$ which follows from Lemma 2.

C o r o l l a r y 1. If $f: X \to \mathbb{R}$ is a solution of (1), then $f(X)$ contains no interval of the form $(a,1)$ with $a < 1$.

C o r o l l a r y 2. If $X = \mathbb{R}$ and $g: X \to \mathbb{R}$ is a solution of (1) having Darboux property, then

$$g(X) \subset [1,+\infty) \quad \text{if} \quad g(0) = 1;$$

or

$$g(X) \subset (-\sqrt{c},c) \quad \text{if} \quad g(0) = 0,$$

where $c \in (0,1)$ and $c^k + c^l = 1$.

Define functions $\varphi_t, \psi_t: \mathbb{R} \to \mathbb{R}$ writing for $s \in \mathbb{R}$

$$\varphi_t(s) = s\, g(t)^k + t\, g(s)^l \tag{2}$$

and

$$\psi_t(s) = t\, g(s)^k + s\, g(t)^l. \tag{3}$$

It is obvious that φ_t and ψ_t both have Darboux property if g has.

P r o p o s i t i o n . Let $X = \mathbb{R}$ and let $g: X \to \mathbb{R}$ be a solution of (1) having Darboux property. Then either $g \equiv 1$ or $g \equiv 0$.

P r o o f . First consider the case $g(0) = 1$. If $g \not\equiv 1$ then, by Corollary 2, $g(t_0) > 1$ for a $t_0 \neq 1$. If $g(t_0)^k \geqslant g(-t_0)^l \geqslant 1$, then

$$\operatorname{sgn} \varphi_{t_0}(0) = \operatorname{sgn} t_0 \neq \operatorname{sgn} t_0\left(g(-t_0)^l - g(t_0)^k\right) = \operatorname{sgn} \varphi_{t_0}(-t_0)$$

so that there exists an $u \neq 0$ between 0 and $-t_0$ such that $\varphi_{t_0}(u) = 0$ which together with (1) gives

$$1 = g(0) = g(\varphi_{t_0}(u)) = g(u)g(t_0) \geqslant g(t_0) > 1.$$

Similarly we obtain a contradiction, if $g(t_0)^k < g(-t_0)^l$. Indeed, then $g(-t_0) > 1$ and (cf. (3))

$$\operatorname{sgn} \psi_{-t_0}(0) = \operatorname{sgn}(-t_0) \neq \operatorname{sgn}(-t_0(g(t_0)^k - g(-t_0)^l)) =$$
$$= \operatorname{sgn} \psi_{-t_0}(t_0)$$

and $\psi_{-t_0}(u) = 0$ for an $u \neq 0$ lying between 0 and t_0 which leads to

$$1 = g(0) = g(\psi_{-t_0}(u)) = g(u)g(-t_0) \geqslant g(-t_0) > 1.$$

Now let us proceed to the case $g(0) = 0$. It follows from Corollary 2 that

$$(4) \qquad V_g := \sup\{|g(t)| : t \in \mathbb{R}\} < 1.$$

As $g(\mathbb{R})$ is a multiplicative semigroup, then it is easy to see that $g \not\equiv 0$ implies existence of a $t \in \mathbb{R}$ such that $g(t) > 0$. As g is bounded we have (cf. (2))

$$\lim_{s \to +\infty} \varphi_t(s) = +\infty, \qquad \lim_{s \to -\infty} \varphi_t(s) = -\infty,$$

hence $\varphi_t(\mathbb{R}) = \mathbb{R}$. Take any $s \in \mathbb{R}$ and choose $u \in \mathbb{R}$ such that $\varphi_t(u) = s$. Then we have

$$|g(s)| = |g(\varphi_t(u))| = |g(t)g(u)| \leqslant V_g^2.$$

Therefore $V_g \leqslant V_g^2$ which together with (4) gives $V_g = 0$ and ends the proof.

Now shall generalize the above Proposition.

T h e o r e m . Let X be a linear space over reals and let $\{e_i\}_{i\in I}$ be its algebraic base. Further let $f: X \to \mathbb{R}$ be a solution of equation (1). Then

(i) if $f(0) = 0$ and for every $x \in X \setminus \{0\}$ function $g_x: \mathbb{R} \to \mathbb{R}$ given by

(5) $$g_x(t) = f(tx)$$

for $t \in \mathbb{R}$ has Darboux property, then $f \equiv 0$;

(ii) if $f(0) = 1$ and for every $i \in I$ function g_{e_i} defined by (5) has Darboux property, then $f \equiv 1$.

Proof. It is straightforward matter to check that g_x defined by (5) is a solution of (1) for every $x \in X \setminus \{0\}$. Of course $g_x(0) = f(0)$ for every $x \in X \setminus \{0\}$. Thus, in case (i), $g_x \equiv 0$, by our Proposition. Suppose now that $f(0) = 1$ and the functions g_{e_i} have Darboux property. Once more using Proposition we obtain that $g_{e_i} \equiv 1$ for $i \in I$. Take arbitrary $x = \alpha_1 e_{i_1} + \dots + \alpha_n e_{i_n} \in X$. We will show by induction with regard to n that $f(x) = 1$. Indeed, for every $\alpha \in \mathbb{R}$ and $i \in I$ we have $f(\alpha e_i) = g_{e_i}(\alpha) = 1$. Suppose that $f(\alpha_1 e_{i_1} + \dots + \alpha_n e_{i_n}) = 1$ for every $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $i_1, \dots, i_n \in I$. Take $y = \beta_1 e_{j_1} + \dots + \beta_{n+1} e_{j_{n+1}}$, where $\beta_1, \dots, \beta_{n+1} \in \mathbb{R}$ and $j_1, \dots, j_{n+1} \in I$. By induction hypothesis we have $f(\beta_1 e_{j_1} + \dots + \beta_n e_{j_n}) = 1$ and $f(\beta_{n+1} e_{j_{n+1}}) = 1$. Thus from (1) it follows that

$$f(y) = f(\beta_1 e_{j_1} + \dots + \beta_{n+1} e_{j_{n+1}}) = f((\beta_1 e_{j_1} + \dots + \beta_n e_{j_n}) \times$$
$$\times f(\beta_{n+1} e_{j_{n+1}})^k + \beta_{n+1} e_{j_{n+1}} f(\beta_1 e_{j_1} + \dots + \beta_n e_{j_n})^l) =$$
$$= f(\beta_{n+1} e_{j_{n+1}}) f(\beta_1 e_{j_1} + \dots + \beta_n e_{j_n}) = 1,$$

which ends the proof of Theorem.

The following example shows that in the case (i) it is necessary to assume that g_x has Darboux property for all $x \in X \setminus \{0\}$.

Example. The function $f: \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = \begin{cases} 0, & x_1 \leqslant 0 \vee x_2 \neq 0, \\ 1, & x_1 > 0 \wedge x_2 = 0 \end{cases}$$

for $(x_1, x_2) \in \mathbb{R}^2$ satisfies (1).

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