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SOME REMARKS ON CAUCHY EQUATION ON A CURVE

1. In recent years a still increasing number of papers have been dealing with so called conditional Cauchy equations. The problem can be briefly stated as follows. Let X and E be some nonempty sets and let $\circ : X\times X \to X$ and $* : E\times E \to E$ be some binary operations. Further, let $Z\subset X\times X$ be a nonempty set. Consider the following functional equation for $f: X \to E$ (usually called Cauchy functional equation)

$$f(x\circ y) = f(x)*f(y). \tag{1}$$

The problem is to find all functions f satisfying (1) for all $x,y \in Z$ or rather to compare the set of these functions with the set of functions satisfying (1) for all $(x,y)\in X\times X$. As stated here, the question is very general and no wonder that usually some additional assumptions are made about X, E, $\circ$, $*$, Z and even the class of functions in which we are looking for solutions of (1). The reader is referred to J. Dhombres [1] and R. Ger [3] and [4] for more details.

One of the most studied situations is the case where $X = (0, +\infty)$, $E = \mathbb{R}$, $\circ$ and $*$ are both usual addition. In the present paper we shall restrict ourselves to this case. Moreover, Z will be a curve in $X\times X$. In other words we are going to deal with the equation

$$f(x(t) + y(t)) = f(x(t)) + f(y(t)) \tag{2}$$

for $t \in (0,+\infty)$, where $x,y:(0,+\infty) \longrightarrow (0,+\infty)$ are some functions. Let us mention here J. Dhombres [1], G.L. Forti [2], W. Jarczyk [7], M. Laczkovich [10] and M.C. Zdun [12] among those who obtained numerous results in this particular area.

Typical results read as follows. Under some assumptions on functions x and y every solution f of (2) which belongs to a prescribed class (for instance f is continuous, measurable or $\lim_{t\to 0} f(t)/t$ exists) has to be additive (or sometimes equal to an additive function almost everywhere in the sense of Lebesgue measure), i.e. f fulfils

$$f(s + t) = f(s) + f(t) \tag{3}$$

for all $s,t \in (0,+\infty)$.

In what follows we shall consider functions $x,y:(0,+\infty) \longrightarrow (0,+\infty)$ which are continuous and such that $x+y$ maps $(0,+\infty)$ homeomorphically onto itself. If we adopt these assumptions then (2) may be equivalently written as

$$f(t) = f(r_1(t)) + f(r_2(t)) \tag{4}$$

for all $t \in (0,+\infty)$, where $r_1 = x \circ (x+y)^{-1}$ and $r_2 = y \circ (x+y)^{-1}$ and hence r_i are continuous, $0 < r_i(t) < t$ for $t > 0$ and $r_1 + r_2 = \mathrm{id}$. Under these assumptions one can prove (cf. for instance Dhombres [1], Forti [2], Zdun [12] and M. Sablik [11]) that every solution f of (4) such that $\lim_{t\to 0} f(t)/t$ exists is linear, i.e. there is a constant $c \in \mathbf{R}$ such that $f(t) = ct$, $t > 0$. We can say that (3) and (4) are equivalent in the class

$$\underset{\sim}{D} = \left\{ f:(0,+\infty) \longrightarrow \mathbf{R} : \lim_{t\to 0} f(t)/t \text{ exists and is finite} \right\} .$$

(it is a well known fact that any solution of (3) belonging to $\underset{\sim}{D}$ has to be linear). Obviously this equivalence holds also in the class

$\underset{\sim}{A} = \underset{\sim}{D} + \underset{\sim}{Ad}$, where $\underset{\sim}{Ad}$ denotes the class of all solutions of (3). One cannot expect equivalence of (3) and (4) in the class of all functions because if we take $r_i = (1/2)\mathrm{id}$ then there exist nonadditive solutions of (4) (even very regular ones, cf. M. Kuczma [8]).

For every $f:(0,+\infty) \longrightarrow \mathbf{R}$ we call the Cauchy difference of f the function $C_f:(0,+\infty)\times(0,+\infty) \longrightarrow \mathbf{R}$ defined by

$$C_f(s,t) = f(s+t) - f(s) - f(t).$$

Consider the following class of functions

$$\underset{\sim}{B} = \left\{ f:(0,+\infty) \longrightarrow \mathbf{R} : \ C_f(s,t) = o(s+t), (s,t) \longrightarrow (0,0) \right\}$$ [1].

Let $f \in \underset{\sim}{A}$ and write $f = g + a$, where $g \in \underset{\sim}{D}$ and $a \in \underset{\sim}{Ad}$. Denote $L = \lim_{t\to 0} g(t)/t$. Then we have for $s,t>0$

$$C_f(s,t)/(s+t) = [(g(s+t)/(s+t))-L] - (s/(s+t))[(g(s)/s)-L] - (t/(s+t))[(g(t)/t)-L],$$

whence it easily follows that $\lim_{(s,t)\to(0,0)} C_f(s,t)/(s+t) = 0$. We have proved therefore that $\underset{\sim}{A} \subset \underset{\sim}{B}$. The reverse inclusion does not hold as is shown by the following example which we owe to Z. Gajda.

Example. Let $h:(0,+\infty) \longrightarrow \mathbf{R}$ be given by $h(t) = t - [t]$ and put

$$f(t) = \sum_{n=1}^{\infty} h(2^n t)/n2^n$$

for every $t>0$. It can be checked that f is continuous, $\lim_{t\to 0} f(t)/t = +\infty$ and $f \in \underset{\sim}{B}$. If we had $f = g + a$ with $g \in \underset{\sim}{D}$ and an additive a then,

[1] Here and in the sequel we use the Landau notation in what concerns functions with limits equal to zero.

in view of continuity of f, a would be linear and hence differentiable at 0. Thus f would be in $\underset{\sim}{D}$, a contradiction.

It is still an open question whether (3) and (4) are equivalent in the class $\underset{\sim}{B}$ for any suitable pair of r_i's. Below we prove a result showing that such an equivalence actually holds if we consider some special curves.

Theorem 1. Let $\underset{\sim}{C}$ be the class of functions defined by

$$\underset{\sim}{C} = \left\{ f:(0,+\infty) \to \mathbf{R} : C_f(t\cos\alpha, t\sin\alpha) = o(t),\ t \to 0, \text{ for all } \alpha \in (0,\pi/2) \right\}.$$

Further, assume that

(H) $r_i:(0,+\infty) \to (0,+\infty)$, $i = 1,2$, are continuous and $r_1+r_2 = \mathrm{id}$.

Then every function $f \in \underset{\sim}{C}$ satisfying (4) and

$$f(2t) = f(2r_1(t)) + f(2r_2(t)) \tag{5}$$

for $t>0$, is additive.

Proof. Define $F:(0,+\ ) \to \mathbf{R}$ by $F(t) = C_f(t,t)$. By (4) and (5) we have for every $t>0$

$$F(t) = f(2t) - 2f(t) = f(2r_1(t)) - 2f(r_1(t)) + f(2r_2(t)) - 2f(r_2(t)) =$$

$$= F(r_1(t)) + F(r_2(t)).$$

Moreover, we have for every $s \in (0,+\infty)$

$$F(s)/s = C_f(s,s)/s = \sqrt{2}\, C_f(\sqrt{2}s/\sqrt{2}, \sqrt{2}s/\sqrt{2})/\sqrt{2}s =$$

$$= \sqrt{2}C_f(\sqrt{2}s\cos(\pi/4), \sqrt{2}s\sin(\pi/4)),$$

whence

$$F(s) = o(s), \quad s \to 0+, \tag{6}$$

because $f \in \underset{\sim}{C}$. Thus F fulfils (4) and (6) which means that $F = 0$ (cf. e.g. Dhombres [1]). Fix now $s,t>0$. We get

$$0 = F(s+t) = f(2s+2t) - 2f(s+t) = C_f(2s,2t) + C_f(s,s) + C_f(t,t) - $$
$$- 2C_f(s,t) = C_f(2s,2t) - 2C_f(s,t)$$

whence

$$C_f(2s,2t) = 2C_f(s,t)$$

for every $s,t > 0$. An easy induction yields

$$(7) \qquad C_f(s,t) = 2^n C_f(s/2^n, t/2^n)$$

for every s,t 0 and $n \in \mathbf{N}$. Fix $s,t>0$ and choose $u>0$ and $\alpha \in (0,\pi/2)$ so that $s = u \cos\alpha$ and $t = u \sin\alpha$. From (7) and our assumptions we infer

$$C_f(s,t) = \lim_{n \to \infty} u[C_f((u/2^n)\cos\alpha, (u/2^n)\sin\alpha)/(u/2^n)] = 0$$

which proves additivity of f.

Corollary 1. Let r_i, $i = 1,2$, satisfy (H) and assume moreover that $r_1(2t) = 2r_1(t)$ for $t>o$. Then every solution of (4) which belongs to $\underset{\sim}{B}$ is additive.

Proof. We get immediately $r_2(2t) = 2r_2(t)$ for $t>0$. This means that every solution of (4) satisfies (5) as well. Since obviously $\underset{\sim}{B} \subset \underset{\sim}{C}$ we get our assertion from Theorem 1.

Observe that assumptions of Corollary 1 are satisfied in particular when $r_1(t) = ct$ for a $c \in (0,1)$.

2. M. Laczkovich in [9] introduced the notion of double difference property which in present circumstances can be formulated as follows. Let $\underset{\sim}{M}_1$ and $\underset{\sim}{M}_2$ be classes of real functions defined in $(0,+\infty)$ and $(0,+\infty)\times(0,+\infty)$, respectively. We say that $(\underset{\sim}{M}_1, \underset{\sim}{M}_2)$ has the double difference property (which we abbreviate to d.d.p. in the sequel) if for every function $f:(0,+\infty) \to \mathbf{R}$ we have

$C_f \in \underset{\sim}{M}_2$ implies $f = a + g$ where a is additive and $g \in \underset{\sim}{M}_1$.

Laczkovich proved that $(\underset{\sim}{L}^1, \underset{\sim}{L}^2)$ where $\underset{\sim}{L}^i$ denotes Lebesgue measurable functions of i variables, $i = 1,2$, has the d.d.p. On the other hand, if $\underset{\sim}{M}_i$, $i = 1,2$, is the class of bounded functions then $(\underset{\sim}{M}_1, \underset{\sim}{M}_2)$ has the d.d.p. This last sentence is known under the name of Hyers theorem on the stability of Cauchy functional equation (cf. [5]).

Let us introduce the following classes of functions for every $p \geq 1$:

$$\underset{\sim}{D}_p^1 = \left\{ f:(0,+\infty) \to \mathbf{R} : f(t) = o(t^p),\ t \to 0 \right\},$$

and

$$\underset{\sim}{D}_p^2 = \left\{ G:(0,+\infty)\times(0,+\infty) \to \mathbf{R} : G(s,t) = o((s+t)^p), (s,t) \to (0,0) \right\}.$$

Gajda's example quoted in § 1 shows that $(\underset{\sim}{D}_1^1, \underset{\sim}{D}_1^2)$ has not the d.d.p. We will show below that $(\underset{\sim}{D}_p^1, \underset{\sim}{D}_p^2)$ has the d.d.p. for every $p > 1$.

We start with a result which could be called d.d.p. on a curve.

L e m m a 1. Let r_i, $i = 1,2$, satisfy (H). Further, let $f:(0,+\infty) \to \mathbf{R}$ be a function and define functions $F_n:(0,+\infty) \to \mathbf{R}$, $n \in \mathbf{N}$, by

$$F_1(t) = C_f(r_1(t), r_2(t)) \quad \text{and} \quad F_{n+1}(t) = F_n(r_1(t)) + F_n(r_2(t)).$$

Assume that for some $p \geq 1$ and $t_o > 0$

$$F_1 \in \underset{\sim}{D}_p^1 \quad \text{and} \quad \sum_{n=1}^{\infty} F_n(t)/t^p \quad \text{is uniformly convergent in } (0,t_o].$$

Then $f = a + g$ where a fulfils (4) and $g \in \underset{\sim}{D}_p^1$.

P r o o f . An easy induction shows that for every $n \in \mathbf{N}$ and $t > 0$

$$(8) \qquad f(t) = \sum_{i_1,\dots,i_n \in \{1,2\}} f(r_{i_1} \circ \dots \circ r_{i_n}(t)) + \sum_{k=1}^{n} F_k(t).$$

Let $h(t) = \sum_{n=1} F_n(t)/t^p$ for $t \in (0,t_o]$ where $p \geq 1$ and $t_o > 0$ are as in our assumptions. Put $g_o(t) = t^p h(t)$ for $t \in (0,t_o]$. From (8) we infer

$$a_o(t) = \lim_{n\to\infty} \sum_{i_1,\dots,i_n\in\{1,2\}} f(r_{i_1}\circ\dots\circ r_{i_n}(t)) = f(t) - g_o(t)$$

exists for every $t\in(0,t_o]$. Moreover, since

$$\sum_{i_1,\dots,i_{n+1}\in\{1,2\}} f(r_{i_1}\circ\dots\circ r_{i_n}(t)) =$$

$$= \sum_{i=1}^{2}\sum_{i_1,\dots,i_n\in\{1,2\}} f(r_{i_1}\circ\dots\circ r_{i_n}\circ r_i(t))$$

for $n\in\mathbf{N}$ and $t>0$ we obtain

(9) $$a_o(t) = a_o(r_1(t)) + a_o(r_2(t))$$

for $t\in(0,t_o]$.

Now, let $n\in\mathbf{N}$ be such that $r_{i_1}\circ\dots\circ r_{i_n}(t)\in(0,t_o]$ for all $i_1,\dots,i_n\in\{1,2\}$ and put

(10) $$a(t) = \sum_{i_1,\dots,i_n\in\{1,2\}} a_o(r_{i_1}\circ\dots\circ r_{i_n}(t)).$$

We show first that the definition (10) is not depending on n. Indeed, let n and m be such that $r_{i_1}\circ\dots\circ r_{i_q}(t)\in(0,t_o]$ for every $i_1,\dots,i_q\in\{1,2\}$ and $q\in\{n,m\}$. Suppose that $n<m$. Since $0<r_i<\mathrm{id}$ and r_i are continuous, $i = 1,2$, we have $r_{i_1}\circ\dots\circ r_{i_{n+k}}(t)\in$ $\in(0,t_o]$ for every $k\in\mathbf{N}$ and $i_1,\dots,i_{n+k}\in\{1,2\}$. In particular we get by (9)

$$\sum_{i_1,\dots,i_m\in\{1,2\}} a_o(r_{i_1}\circ \dots \circ r_{i_m}(t)) =$$

$$= \sum_{i_2,\dots,i_m\in\{1,2\}} a_o(r_{i_2}\circ \dots \circ r_{i_m}(t)) =$$

$$= \sum_{i_1,\dots,i_{m-1}\in\{1,2\}} a_o(r_{i_1}\circ \dots \circ r_{i_{m-1}}(t)).$$

An easy induction leads to

$$\sum_{i_1,\dots,i_m\in\{1,2\}} a_o(r_{i_1}\circ \dots \circ r_{i_m}(t)) =$$

$$= \sum_{i_1,\dots,i_n\in\{1,2\}} a_o(r_{i_1}\circ \dots \circ r_{i_n}(t))$$

which proves that the definition is correct. A very similar argument shows that the function a defined by (10) for $t > t_o$ and equal to a_o in $(0,t_o]$ fulfils (4).

Define $g:(0,+\infty) \to \mathbf{R}$ by $g(t) = f(t) - a(t)$. It remains to prove that $g(t) = o(t^p)$, $t \to 0$. We have $g|(0,t_o] = g_o$. Moreover,

$$F_n(t) = o(t^p), \quad t \to 0, \tag{11}$$

for every $n \in \mathbf{N}$. Indeed, we assumed it for $n = 1$ and if (11) holds for an $n \in \mathbf{N}$ then from the equality

$$F_{n+1}(t)/t^p = \sum_{i=1}^{2} (r_i(t)/t)^p (F_n(r_i(t))/t^p),$$

which is derived from the definition of F_{n+1}, we easily get the validity of (11) for n+1 as well. Combining (11) with the uniform convergence of $\sum_{n=1}^{\infty} F_n(t)/t^p$ we see that $g \in \underset{\sim}{D}_p^1$ which ends the proof.

Taking into account Corollary 1 we get from the above Lemma

Corollary 2. Under the assumptions of Lemma 1, if moreover $C_f \in \underset{\sim}{D}_1^2$ and $r_1(2t) = 2r_1(t)$ for every $t>0$ then $f = a + g$ where a is additive and $g \in \underset{\sim}{D}_1^1$.

Proof. By Lemma 1 the function a fulfils (4). Moreover, $g \in \underset{\sim}{D}_1^1 \subset \underset{\sim}{D} \subset \underset{\sim}{A} \subset \underset{\sim}{B}$. We also have $f \in \underset{\sim}{B}$ and hence obviously $a = f - g \in \underset{\sim}{B}$. Thus a is additive by Corollary 1.

Let us prove now that $(\underset{\sim}{D}_p^1, \underset{\sim}{D}_p^2)$ has the d.d.p. for $p>1$.

Theorem 2. Let $f:(0,+\infty) \to R$ be a function such that $C_f \in \underset{\sim}{D}_p^2$ for some $p>1$. Then $f = a + g$ where a is additive and $g \in \underset{\sim}{D}_p^1$.

Proof. Define $r_i:(0,+\infty) \to R$ by $r_i(t) = t/2$ for $i = 1,2$. Using the notation of Lemma 1 we get by an easy induction

$$(12) \qquad F_1(t) = C_f(t/2,t/2) \text{ and } F_{n+1}(t) = 2^n F_1(t/2^n) \text{ for } n \geq 1.$$

Obviously we have $F_1(t) = o(t^p)$, $t \to 0+$, because $C_f \in \underset{\sim}{D}_p^2$. It follows from (12) that

$$(13) \qquad F_{n+1}(t)/t^p = (2^n)^{(1-p)}[F_1(t/2^n)/(t/2^n)^p], \quad n \geq 1.$$

Thus $F_{n+1}(t) = o(t^p)$, $t \to 0$, for every $n \geq 1$. Choose $t_o > 0$ so that

$$|F_1(t)/t^p| \leq M$$

for a constant $M \geq 0$ and all $t \in (0,t_o]$. Then in view of (13) we have

$$|F_{n+1}(t)/t^p| \leq M(2^{1-p})^n$$

for every $t \in (0,t_o]$ and $n \geq 1$. This proves that the series $\sum_{n=1}^{\infty} F_n(t)/t^p$ is uniformly convergent in $(0,t_o]$. By Lemma 1 we get the decomposition $f = a + g$, where $g(t) = o(t^p)$, $t \to 0$, and

$$a(t) = 2a(t/2)$$

for every $t>0$. Now, if $C_f \in \underset{\sim}{D}_p^2$ for a $p>1$ then $C_f \in \underset{\sim}{D}_1^2$ and hence $f \in \underset{\sim}{B}$. Now the assertion follows from Corollary 2.

3. Theory of stability of functional equations was born in 1940s with papers of D.H. Hyers and S. Ulam (cf. [5] and [6] for instance). It has much developed since and even a short presentation of its achievements would go far beyond the frame of the present paper. As we have mentioned in the previous section, the stability of Cauchy equation in the sense of Hyers can be looked at as d.d.p. problem. On the other hand those who prefer stability language would rather say that d.d.p problems belong in fact to stability theory - one just replaces boundedness of Cauchy difference by some other regularity property. From this point of view the results of the previous section are also of "stability type" and Lemma 1 cound be called a result on stability on a curve.

In Theorem 2, to prove additivity of a we have used its additivity on the diagonal and the fact that a has some regularity at 0. We could not expect that a similar method would work if we assumed only the boundedness of a. In fact the equation

$$a(t) = 2a(t/2)$$

has many solutions which are bounded and continuous in $(0,+\infty)$, still not additive. Thus one cannot hope that the assertion of Hyers theorem will hold if we assume that C_f is bounded on curve only. To get such an assertion one has to impose some other conditions on f. Below we present a possible approach to this problem.

Let us prove first

T h e o r e m 3. Let $f:(0,+\infty) \to [0,+\infty)$ be Lebesgue measurable and such that

(14) $$|f(t) - 2f(t/2)| \leq \varepsilon , \quad t > 0$$

and

(15) function $t \to f(t) - f(ct) - f(dt)$ is bounded in $(0,\infty)$

where $\varepsilon \geq 0$ is a constant, $c \in (0,1)$, $d = 1 - c$ and $\ln c/\ln d \notin \mathbf{Q}$. Then $f = a + g$ where a is equal to an additive function almost everywhere, g is measurable and $|g| \leq \varepsilon$.

Proof. Observe that (14) is equivalent to $|f(2t) - 2f(t)| \leq \varepsilon$ which we write as

(16) $$|C_f(t,t)| \leq \varepsilon$$

for every $t > 0$. An easy induction shows that

(17) $$f(t) = (1/2)f(2t) - (1/2)C_f(t,t) = \dots$$
$$= (1/2^n)f(2^n t) - \sum_{k=1}^{n} (1/2^k)C_f(2^k t, 2^k t).$$

From (16) we infer that the series $\sum_{n=1}^{\infty} (1/2^n)C_f(2^n t, 2^n t)$ is convergent. Define $g: (0,+\infty) \to \mathbf{R}$ by

$$g(t) = - \sum_{n=1}^{\infty} (1/2^n)C_f(2^n t, 2^n t).$$

Then g is measurable because so is $t \mapsto C_f(t,t)$, and (cf. (16)) $|g(t)| \leq \varepsilon$ for every $t > 0$. From (15) we infer that

$$a(t) = \lim_{n\to\infty} (1/2^n)f(2^n t)$$

exists for every $t > 0$. It is easy to observe that a is nonnegative, measurable and $a(2t) = 2a(t)$. Furthermore we have by (15) for every $n \in \mathbf{N}$ and $t > 0$ and some $K > 0$

$$(1/2^n)[f(2^n t) - f(2^n ct) - f(2^n dt)] \in (1/2^n)[-K, K]$$

whence, letting $n \to \infty$ we obtain

$$a(t) = a(ct) + a(dt).$$

Taking into account Laczkovich's result from [10] we see that a is a.e. equal to an additive function, which ends the proof.

The last result of the paper reads as follows.

Theorem 4. Let $f:(0,+\infty) \to \mathbb{R}$ be a function fulfilling (14) with an $\varepsilon \geqslant 0$ and suppose that an $n \in \mathbb{N}$ exists such that function

(18) $$t \to f(t) - f(t/2^N) - f((2^N-1)t/2^N)|$$

or

(19) $$t \to f(t) - f(t/(2^N+1)) - f(2^N t/(2^N+1))$$

is bounded in $(0,+\infty)$. If f is measurable then the assertion of Theorem 3 is valid.

Proof. Similarly as in the proof of Theorem 3 we get $f = a + g$ where a and g are measurable, $|g| \leqslant \varepsilon$, $a(t) = 2a(t/2)$ for $t > 0$ and

(20) $$a(t) = a(t/2^N) + a((2^N-1)t/2^N)$$

or

(21) $$a(t) = a(t/(2^N+1)) + a(2^N t/(2^N+1))$$

depending on whether (18) or (19) is assumed. We easily check that

(22) $$a(2^k t) = 2^k a(t) \quad \text{for all } k \in \mathbb{Z} \text{ and } t > 0$$

and hence also

(23) $$a((2^N-1)^m t) = (2^N-1)^m a(t) \quad \text{for all } m \in \mathbb{Z} \text{ and } t > 0,$$

if (20) holds, or

(24) $a((2^N+1)^m t) = (2^N+1)^m a(t)$ for all $m \in \mathbf{Z}$ and $t>0$,

if (21) is valid. In both cases we obtain

(25) $$a(rt) = ra(t)$$

for every $t>0$ and $r \in \underset{\sim}{G}$ where $\underset{\sim}{G} = \left\{2^k(2^N-1)^m\colon n,m \in \mathbf{Z}\right\}$ in the case (22), (23), and $\underset{\sim}{G} = \left\{2^k(2^N+1)^m\colon n,m \in \mathbf{Z}\right\}$ in the case (22), (24). Since neither $\ln 2/\ln(2^N-1)$ nor $\ln 2/\ln(2^N+1)$ is a rational number, the set $\underset{\sim}{G}$ is dense in $(0,+\infty)$. This implies that $\ln \underset{\sim}{G}$ is dense in $\mathbf{R}$ whence we infer that the function $b\colon \mathbf{R} \to \mathbf{R}$ given by

$$b(v) = a(\exp v)/\exp v$$

is microperiodic. Since b is measurable, it has to be constant almost everywhere. Thus there is a $c \in \mathbf{R}$ such that

$$a(\exp v) = c\exp v$$

for almost all $v \in \mathbf{R}$. Absolute continuity of exp implies that $a(t) = ct$ a.e. in $(0,+\infty)$ which ends the proof.

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