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Author: Władysław Kierat, Urszula Sztaba

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Władysław Kierat, Urszula Sztaba

THE GENERALIZED FOURIER TRANSFORMS OF SLOWLY INCREASING FUNCTIONS

In the paper we show that

$$\widehat{\mathcal{F}}f = (\mathcal{C} \circ \mathcal{F})f$$

for slowly increasing function f, where the symbols $\widehat{\mathcal{F}}f, \mathcal{C}f$ and $\mathcal{F}f$ denote the generalized Fourier, Cauchy and Fourier transforms of f, respectively.

1. The Cauchy transform of Λ in D'_{L^p}

Let D_{L^q} denote the set of all smooth functions φ such that $\varphi^{(k)} \in L^q(\mathbf{R})$ for $k = 0, 1, \ldots$, where $1 < q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p < \infty$. The space D_{L^q} is equipped with a topology defined by the sequence $(\|\cdot\|_k)$ of the norms

$$\|\varphi\|_k := \max_{i=0,\ldots k} \|\varphi^{(i)}\|_{L^q}.$$

We shall denote by D'_{L^p} the space of all linear continuous forms on D_{L^q}.

Let $\Lambda \in D'_{L^p}$. We take

$$\mathcal{C}\Lambda(z) := \frac{1}{2\pi i}\Lambda\left(\frac{1}{\cdot - z}\right) \qquad \text{for} \quad z \in \mathbb{C} - \mathbf{R}, \tag{1}$$

where $\mathcal{C}\Lambda$ is called the Cauchy transform of Λ.

The Cauchy transforms of the distributions belonging to D'_{L^p} have important properties. Namely, the following theorems hold.

THEOREM 1 ([3]). *If $\Lambda \in D'_{L^p}$, then the Cauchy transform $\mathcal{C}\Lambda$ is a holomorphic function in $\mathbb{C} - \mathbf{R}$ and*

$$\frac{d^k}{dz^k}\mathcal{C}\Lambda(z) = \frac{k!}{2\pi i}\Lambda\left(\frac{1}{(\cdot - z)^{k+1}}\right) = \mathcal{C}\Lambda^{(k)}(z).$$

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THEOREM 2 ([3]). *If* $\Lambda \in D'_{L^p}$, $1 < p < \infty$, *then*

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} [C\Lambda(x + i\varepsilon) - C\Lambda(x - i\varepsilon)]\varphi(x)\,dx = \Lambda(\varphi) \tag{2}$$

for $\varphi \in D_{L^q}$, $\frac{1}{p} + \frac{1}{q} = 1$.

2. The Fourier transform of Λ in D'_{L^2}

By Schwartz representation theorem ([4],p.201) we have

$$\Lambda(\varphi) = \sum_{\nu=0}^{k}(-1)^\nu \int_{-\infty}^{\infty} f_\nu(x)\varphi^{(\nu)}(x)\,dx, \tag{3}$$

where $f_\nu \in L^2(\mathbf{R})$ and $\varphi \in \mathcal{S}$ with $\mathcal{S}$ denoting the space of rapidly decreasing functions.

Let $\mathcal{F}\Lambda$ and $\mathcal{F}\varphi$ denote the Fourier transforms of Λ and φ, respectively. Since $\mathcal{F}\Lambda(\varphi) = \Lambda(\mathcal{F}\varphi)$, by Parseval-Plancherel's formula, we obtain

$$\mathcal{F}\Lambda(\varphi) = \sum_{\nu=0}^{k} \int_{-\infty}^{\infty} (ix)^\nu \mathcal{F}f_\nu(x)\varphi(x)\,dx \qquad \text{for } \varphi \in \mathcal{S}.$$

This equality can be written as follows

$$\mathcal{F}\Lambda(x) = \sum_{\nu=0}^{k}(ix)^\nu g_\nu(x), \qquad \text{where } g_\nu = \mathcal{F}f_\nu. \tag{4}$$

Of course g_ν is in $L^2(\mathbf{R})$.

Denote by $\mathcal{S}'_0$ the vector space spanned by the set of all functions $x^\nu\varphi$, where φ is in $L^2(\mathbf{R})$. These functions are called slowly increasing. Clearly, if f is in $\mathcal{S}'_0$, then $\mathcal{F}f$ is in D'_{L^2} (see [4]). In general, there is no any Fourier transform of $f \in \mathcal{S}'_0$ in the classical sense.

3. The generalized Fourier transform

T. Carleman defined in [2] the generalized Fourier transforms of slowly increasing functions as follows.

DEFINITION 1. For $f \in \mathcal{S}'_0$ we have

$$\widehat{\mathcal{F}}f(z) = \begin{cases} \int_0^\infty f(t)e^{itz}\,dt & \text{for } \operatorname{Im} z > 0, \\ -\int_{-\infty}^0 f(t)e^{itz}\,dt & \text{for } \operatorname{Im} z < 0, \end{cases}$$

and

$$\widehat{\mathcal{F}}^{-1}f(z) = \begin{cases} \frac{1}{2\pi}\int_{-\infty}^0 f(t)e^{-itz}\,dt & \text{for } \operatorname{Im} z > 0, \\ -\frac{1}{2\pi}\int_0^\infty f(t)e^{-itz}\,dt & \text{for } \operatorname{Im} z < 0. \end{cases}$$

Now we prove the following theorem.

THEOREM 3. *If f is in S_0', then*

$$\widehat{\mathcal{F}}f(z) = (C \circ \mathcal{F})f(z) \quad \text{for} \quad z \in \mathbb{C} - \mathbb{R}, \tag{5}$$

where $\mathcal{F}f$ is the Fourier transform of f in the distributional sense.

Proof. We first prove a reduced form of the theorem. Namely, we assume that f is in $L^2(\mathbb{R})$. Determine $(C \circ \mathcal{F})f(z)$ for $z \in \mathbb{C} - \mathbb{R}$. According to the definitions of $\mathcal{F}$ and C, we have

$$(C \circ \mathcal{F})f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{F}f(t) \frac{1}{t - z}\, dt. \tag{6}$$

Applying Parseval–Plancherel's formula, we get

$$(C \circ \mathcal{F})f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(t) \mathcal{F}\frac{1}{(\cdot - z)}(t)\, dt.$$

It is known ([1], p.174) that

$$\frac{1}{2\pi i}\mathcal{F}\frac{1}{(\cdot - z)}(t) = \begin{cases} H(t)e^{itz} & \text{for } \operatorname{Im} z > 0, \\ -H(-t)e^{itz} & \text{for } \operatorname{Im} z < 0, \end{cases}$$

where H denotes the Heaviside step function. From this we obtain

$$(C \circ \mathcal{F})f(z) = \begin{cases} \int_0^{\infty} e^{itz} f(t)\, dt & \text{for } \operatorname{Im} z > 0, \\ -\int_{-\infty}^{0} e^{itz} f(t)\, dt & \text{for } \operatorname{Im} z < 0. \end{cases}$$

Finally we have

$$(C \circ \mathcal{F})f(z) = \widehat{\mathcal{F}}f(z) \quad \text{for } f \in L^2(\mathbb{R}).$$

We are now in a position to show that the theorem is true in general case. It remains to prove that the equality (5) holds for $f(x) = (ix)^k f_0(x)$, where $f_0 \in L^2(\mathbb{R})$ and $k \in N$. In fact, we have

$$(C \circ \mathcal{F})[(i(\cdot))^k f_0(\cdot)](z) = C(\mathcal{F}f_0)^{(k)}(z) = [(C \circ \mathcal{F})f_0]^{(k)}(z) = \frac{d^k}{dz^k}\widehat{\mathcal{F}}f_0(z) =$$

$$= \frac{d^k}{dz^k} \left\{ \begin{array}{ll} \int\limits_0^{\infty} e^{itz} f_0(t)\, dt & \text{for } \operatorname{Im} z > 0 \\ -\int\limits_{-\infty}^{0} e^{itz} f_0(t)\, dt & \text{for } \operatorname{Im} z < 0 \end{array} \right\} =$$

$$
= \left\{ \begin{array}{ll} \int\limits_0^\infty e^{itz}(it)^k f_0(t)\,dt & \text{for } \operatorname{Im} z > 0 \\ \\ -\int\limits_{-\infty}^0 e^{itz}(it)^k f_0(t)\,dt & \text{for } \operatorname{Im} z < 0 \end{array} \right\} = \widehat{\mathcal{F}}f(z).
$$

Hence, by the definition of S_0' we have

$$
(\mathcal{C}\circ\mathcal{F})f(z) = \widehat{\mathcal{F}}f(z) \quad \text{for all} \quad f \in \mathcal{S}_0'.
$$

As a corollary from the previous theorems, we obtain the following one (see [1],§8.16,p.87).

THEOREM 4. *If $f \in S_0'$, then $\widehat{\mathcal{F}}f$ is a holomorphic function in $\mathbb{C}-\mathbb{R}$ and for $\varphi \in D_{L^2}$ we have*

$$
\lim_{\varepsilon\to 0} \int\limits_{-\infty}^{\infty} \left[\widehat{\mathcal{F}}f(x+i\varepsilon) - \widehat{\mathcal{F}}f(x-i\varepsilon)\right]\varphi(x)\,dx = (\mathcal{F}f)(\varphi).
$$

In the same way one can prove the following equality

$$
\widehat{\mathcal{F}}^{-1}f(z) = (\mathcal{C}\circ\mathcal{F}^{-1})f(z) \quad \text{for} \quad f \in \mathcal{S}_0'.
$$

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INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
ul. Bankowa 14
40-007 KATOWICE, POLAND

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