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Author: Erwin Kasparek

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Erwin Kasparek

THE HOMOMORPHISMS OF THE PSEUDO-ORTHOGONAL GROUP OF THE INDEX ONE INTO AN ABELIAN GROUP

In the paper [1] there has been determined all the homomorphisms of a pseudo- orthogonal subgroup GL(2,R) into the multiplication group of reals numbers. In this paper we will solve this problem for a pseudo-orthogonal subgroup of the index one of the group GL(n,R).

Denote by E_1 the diagonal matrix of the form

$$E_1 = \begin{bmatrix} 1 & & & \\ & \cdot & & 0 \\ & & \cdot & \\ & 0 & & 1 & \\ & & & & -1 \end{bmatrix}$$

and the pseudo-orthogonal subgroup $O_1(n,R)$ of the index one we define as follows

$$O_1(n,R) = \{A \in GL(n,R) \; : \; A^T E_1 A = E_1\} \; .$$

It is easy to show the following.

L e m m a 1. Let G be an arbitrary group and N its normal subgroup. If there is a subgroup $H \subset G$ such that

(i) $N \cap H = \{e\}$, where e is the neutral element of G ,

(ii) for each element $g \in G$ there are $n \in N$ and $h \in H$ such that $g = nh$ $(g = hn)$, then the function $\varphi : G \longrightarrow H$ determined by (ii) is a homomorphism.

T h e o r e m 1. Let G be an arbitrary group and K its commutator group. If there is a subgroup $H \subset G$ such that the conditions (i) and (ii) of Lemma 1 are satisfied (for $N = K$), then f is a homomorphism of the group G into an abelian group $\mathcal{A}$ if and only if there is a homomorphism $\tilde{f} : H \longrightarrow \mathcal{A}$ such that $f = \tilde{f}\circ\varphi$, where φ is the homomorphism defined in Lemma 1.

P r o o f. (Necessity). Let $g \in G$ and $g = kh$, $k \in K$, $h \in H$ be the unique decomposition of g. Then $f(g) = f(k)f(h)$, but f(k) is equal to the neutral element of the group $\mathcal{A}$, thus

(0) $$f(g) = f(h) .$$

Denote by $\tilde{f}$ the restriction of the f to the subgroup H. Finally, from (0) we have $f = \tilde{f}\circ\varphi$.

(Sufficiency). It is not hard to show, that if $\tilde{f} : H \longrightarrow \mathcal{A}$ is a homomorphism then the function $f : G \longrightarrow \mathcal{A}$, where $f = \tilde{f}\circ\varphi$ is a homomorphism too.

To solve our problem we shall find the commutator group and the subgroup H of the group $O_1(n,R)$ such that the cionditions (i) and (ii) of Lemma 1 are satisfied. Then by Theorem 1 it suffices to determine all the homomorphisms of the subgroup H into an abelian group.

L e m m a 2. For each matrix $A = (a_{ij})$ of the group $O_1(n,R)$ we have :

(a) the function $\alpha : O_1(n,R) \longrightarrow \{-1,1\}$ determined by $\alpha(A) = \operatorname{sgn} a_{nn}$ is a homomorphism ,

(b) if the diagonal matrix

$$\begin{bmatrix} \operatorname{sgn} \frac{\det A}{a_{nn}} & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & \operatorname{sgn} a_{nn} \end{bmatrix}$$

does not coincide with identity matrix then it does not belong to the commutator group of the group $O_1(n,R)$.

P r o o f. (a) Let $A = (a_{ij})$, $B = (b_{ij})$ be two arbitrary matrices of $O_1(n,R)$ and $C = AB$ with the elements c_{ij}.

Then $c_{nn} = \sum_{s=1}^{n-1} a_{ns} b_{sn} + a_{nn} b_{nn}$. Using the Cauchy-Buniakowski inequality we get

$$\left| \sum_{s=1}^{n-1} a_{ns} b_{sn} \right| \le \sqrt{\sum_{s=1}^{n-1} a_{ns}^2} \sqrt{\sum_{s=1}^{n-1} b_{sn}^2} .$$

But

$$\sum_{s=1}^{n-1} a_{ns}^2 = a_{nn}^2 - 1 \quad \text{and} \quad \sum_{s=1}^{n-1} b_{sn}^2 = b_{nn}^2 - 1 ,$$

thus

$$\left| \sum_{s=1}^{n-1} a_{ns} b_{sn} \right| < |a_{nn} b_{nn}| .$$

Finally, we have

$$a_{nn} b_{nn} - |a_{nn} b_{nn}| < c_{nn} < a_{nn} b_{nn} + |a_{nn} b_{nn}| ,$$

thus

1) if $a_{nn}b_{nn} > 0$ then $c_{nn} > 0$

2) if $a_{nn}b_{nn} < 0$ then $c_{nn} < 0$.

Now it is obvious that (a) holds.

(b) If $A \in O_1(n,R)$ then $A^{-1} = E_1 A^T E_1$. Thus the element $\tilde{a}_{nn}$ of the matrix A^{-1} is equal to the element a_{nn} of the matrix A. Therefore if $C = (c_{ij})$ is the commutator of A and B, then by (a) we have $c_{nn} > 0$.

Now if C is the finite product of the commutators then by (a) and by the fact that $\det C = 1$ we conclude that (b) is true.

T h e o r e m 2. For each matrix $A \in O_1(n,R)$ the product

$$\begin{bmatrix} \operatorname{sgn}\dfrac{\det A}{a_{nn}} & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & \operatorname{sgn} a_{nn} \end{bmatrix} A$$

belongs to the commutator group of the group $O_1(n,R)$.

P r o o f. Any matrix $A \in O_1(n,R)$ can be written in the so-called block form. Namely

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & a_{nn} \end{bmatrix},$$

where A_{11} is a square $(n-1)\times(n-1)$ matrix. The inverse matrix A^{-1} has the form

$$A^{-1} = \begin{bmatrix} A_{11}^T & -A_{21}^T \\ -A_{12}^T & a_{nn} \end{bmatrix} .$$

On the other hand, the element a_{nn} of A^{-1} is equal to the algebraic adjunct of a_{nn} divided by the determinant of A. Since $a_{nn} \neq 0$ therefore A_{11} is nonsingular. Thus there are two orthogonal matrices ϑ_1 and ϑ_2 such that $D_{11} = \vartheta_1^T A_{11} \vartheta_2$ is the diagonal matrix, whose elements are the square roots of the eigenvalues of the matrix $A_{11}A_{11}^T$ and det $\vartheta_1 = 1$.
Thus we have

$$(1) \qquad \begin{bmatrix} \vartheta_1^T & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & a_{nn} \end{bmatrix} \begin{bmatrix} \vartheta_2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} .$$

The right-side of (1) is an element of the group $O_1(n,R)$. Therefore in the matrices D_{12} and D_{21} at most one element is different from zero. If however the element d_{in} of the matrix D_{12} is equal to zero, then the element d_{ni} of the matrix D_{21} is equal to zero too, and the element d_{ii} of the matrix D_{11} satisfies the condition $d_{ii} = 1$.
Thus there is a permutation matrix P such that

$$P^T D_{11} P = \begin{bmatrix} 1 & & & \\ & \ddots & & 0 \\ 0 & & 1 & \\ & & & \sqrt{\lambda} \end{bmatrix}$$

and det P = 1. Now we can write

$$(2)\quad \begin{bmatrix} P^T & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 \ \sqrt{\lambda} & x \\ \hline 0 & 0 & y & a_{nn} \end{array}\right]$$

where

$\lambda-x^2= 1$, $\lambda-y^2 = 1$, $x^2-a_{nn}^2 = -1$, $y^2-a_{nn}^2 = -1$, $y\sqrt{\lambda}-x\,a_{nn} = 0$.

From these equations we get

$$y = x \operatorname{sgn} a_{nn} \quad \text{and} \quad \sqrt{\lambda} = |a_{nn}| .$$

Put

$$(3)\qquad \varepsilon = \operatorname{sgn} a_{nn} .$$

From this and (1), (2) we get

$$(4)\quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \varepsilon \end{bmatrix} \begin{bmatrix} \tilde{\vartheta}_1 & 0 \\ 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ & 1 & & 0 \\ & & |a_{nn}| & x \\ \hline 0\ldots0 & & x & |a_{nn}| \end{array}\right] \begin{bmatrix} \tilde{\vartheta}_2 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\tilde{\vartheta}_1 = \vartheta_1 P$ and $\tilde{\vartheta}_2 = P^T\vartheta_2^T$. Since $a_{nn}^2 - x^2 = 1$ and det $\tilde{\vartheta}_1 = 1$ thus det A = ε det $\tilde{\vartheta}_2$.

Now from (3) and from the fact that $(\det A)^2 = 1$ we conclude that

$$\det \tilde{\vartheta}_2 = \operatorname{sgn} \frac{\det A}{a_{nn}} .$$

Put

$$(5)\qquad \eta = \operatorname{sgn} \frac{\det A}{a_{nn}} .$$

Finally the condition (4) can be written in the form

$$(6) \qquad A = \begin{bmatrix} \eta_1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \varepsilon \end{bmatrix} \begin{bmatrix} \vartheta_1^* & 0 \\ 0 & 1 \end{bmatrix} \left[\begin{array}{cccc|c} 1 & & & 0 & 0 \\ & \ddots & & & \vdots \\ & & 1 & & 0 \\ 0 & & & |a_{nn}| & x \\ \hline 0 & \ldots & 0 & x & |a_{nn}| \end{array}\right] \begin{bmatrix} \vartheta_2^* & 0 \\ 0 & 1 \end{bmatrix}$$

where

$$\vartheta_1^* = \begin{bmatrix} \eta & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \tilde{\vartheta}_1 \begin{bmatrix} \eta & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \quad \text{and} \quad \vartheta_2^* = \begin{bmatrix} \eta & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \tilde{\vartheta}_2 .$$

Since $\det \vartheta_1^* = \det \vartheta_2^* = 1$ therefore the orthogonal matrices

$$\begin{bmatrix} \vartheta_1^* & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \vartheta_2^* & 0 \\ 0 & 1 \end{bmatrix}$$

are the elements of the commutator group of the group $O_1(n,R)$. We will show that the matrix

$$\left[\begin{array}{cccc|c} 1 & & & 0 & 0 \\ & \ddots & & & \vdots \\ & & 1 & & 0 \\ 0 & & & |a_{nn}| & x \\ \hline 0 & \ldots & 0 & x & |a_{nn}| \end{array}\right]$$

belongs to the commutator group of the group $O_1(n,R)$ too. Indeed, let

$$|a_{nn}| = \operatorname{ch} t = \frac{e^t + e^{-t}}{2}$$

and

$$x = \mathrm{sht} = \frac{e^t - e^{-t}}{2}.$$

It is easy to verify that

$$\begin{bmatrix} \mathrm{cht}, & \mathrm{sht} \\ \mathrm{sht}, & \mathrm{cht} \end{bmatrix} = \begin{bmatrix} \mathrm{ch}\,\frac{t}{2}, & \mathrm{sh}\,\frac{t}{2} \\ \mathrm{sh}\,\frac{t}{2}, & \mathrm{ch}\,\frac{t}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \mathrm{ch}\,\frac{t}{2}, & \mathrm{sh}\,\frac{t}{2} \\ \mathrm{sh}\,\frac{t}{2}, & \mathrm{ch}\,\frac{t}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1}.$$

From this and from (6) we conclude that the theorem is proved.

Put

$$E = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix}, \quad E^1 = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}, \quad E_1^1 = \begin{bmatrix} -1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & -1 \end{bmatrix}.$$

C o r o l l a r y. The subgroup $H = \{E, E^1, E_1, E_1^1\}$ and the commutator group of the group $O_1(n,R)$ satisfy the conditions (i) and (ii) of Lemma 1. Thus the function $\varphi : O_1(n,R) \longrightarrow H$ determined by

$$(7) \qquad \varphi(A) = \begin{bmatrix} \mathrm{sgn}\,\frac{\det A}{a_{nn}} & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & \mathrm{sgn}\,a_{nn} \end{bmatrix}$$

is a homomorphism.

T h e o r e m 3. A function $f : O_1(n,R) \longrightarrow \mathcal{A}$ is a homomorphism into the abelian group $\mathcal{A}$ if and only if there exist elements $a, b \in \mathcal{A}$ such that $a^2 = b^2 = e$ and

$$f(A) = e\,, \quad \text{if } \varphi(A) = E\,,$$

$$f(A) = a\,, \quad \text{if } \varphi(A) = E\,,$$

$$f(A) = b\,, \quad \text{if } \varphi(A) = E^1\,,$$

$$f(A) = ab, \quad \text{if } \varphi(A) = E^1\,.$$

P r o o f. By Theorem 1 we have

$$f = \tilde{f}\circ\varphi\,, \tag{8}$$

where $\tilde{f} : H \longrightarrow \mathcal{A}$ is a homomorphism and φ is determined by (7). It is easy to show that any homomorphism $\tilde{f} : H \longrightarrow \mathcal{A}$ has the form

$\tilde{f}(E) = e$, $\tilde{f}(E_1) = a$, $\tilde{f}(E^1) = b$, $\tilde{f}(E^1_1) = ab$, where $a^2 = b^2 = e$ and conversely. From this and (8) we conclude that the theorem is proved.

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INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY,
40-007 KATOWICE, POLAND
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