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ON LINDENBAUM'S EXTENSIONS (Part A.)

An extension version of this abstract will appear in Reports on Mathematical Logic.

1. Consider the well-known Lindenbaum lemma:

If X is a consistent set of formulas then there exist consistent and complete set Y such that $X \subseteq Y$.

It is fairly obvious that the Lindenbaum's extension Y is not uniquely determined (cf. [1], [2], [3], [4], [6]). However Tarski has proved in [8], a theorem concerning the power of the class of Lindenbaum's extensions:

(1.1) (Tarski): If $\{p \to (q \to p), p \to [(p \to q) \to q], (q \to s) \to [(p \to q) \to (p \to s)]\} = A$ and $A \subseteq X$ then the only Cn^* – consistent and Cn^* – complete extension of the consistent set X is the class of all two-valued implicational tautologies (Z_2). (Cn_* is the consequence operation based only on the modus ponens and substitution rules).

Later this Tarski's problem was considered in regard to another systems (cf. [2], [6], [7]). In the present paper we shall prove, among others, the existence of the weakest propositional calculus for which the class of all two-valued tautologies is the only consistent and complete extension and we shall generalize the above mentioned Tarski's theorem. Problems considered in this paper were formulated by Professor W. A. Pogorzelski.

Let S be a set of well-formed formulas built by means of propositional variables and some of the connectives $\to, +, *, \equiv, \sim$. R is a set of rules of inference, $Cn(R, X)$ is the standard consequence-operation. R_{0*} denotes the set $\{r_0, r_*\}$ (r_0 – the modus ponens rule, r_* – the substitution rule), 0 is the empty set, Z_2 is the set of all two-valued tautologies. The set

$L(Cn(R,X))$ of all Lindenbaum extensions of the set $Cn(R,X)$ will be defined as follows:

(1.2) $L(Cn(R,X)) = \{Y \subseteq S : Cn(R,X) \subseteq Y = Cn(R,Y) \neq S \wedge \forall_{\Phi \in S-Y} Cn(R, Y \cup \{\Phi\}) = S$.

The following definition is a transformation of Tarski's problem for any $N \not\subseteq S$.

(1.3) $\langle R, X\rangle \in T_M \equiv \forall_{Y \in L(Cn(R,X))} Y = M$.

We assume that the system $\langle R', A'\rangle$ is the M-weakest propositional calculus with the Tarski's property iff the following conditions are satisfied:

(a) $\langle R', A'\rangle \in T_M$
(b) $\langle R, A\rangle \in T_M \Rightarrow R' \subseteq Der(R,A) \wedge A' \subseteq Cn(R,A)$, for every set of rules R and for every $A \subseteq S$.

Let $\langle R, A\rangle \in Cns$ mean, that $Cn(R,S) \neq S$. The notion of T_M can be characterized by the following lemma:

(1.4) $\langle R, A\rangle \in T_M \equiv \forall_{\Phi \in M} Cn(R, A \cup \{\Phi\}) = S \wedge R(M)$, for every $M \subsetneq S$ and for every $\langle R, A\rangle \in Cns$. ($R(M)$ means that M is closed in respect of the rules belonging to the set R).

Let $M \subsetneq S$ and let

(1.5) $\Phi_1, \Phi_2, \ldots$

be the sequence of all formulas from $S - M$. We define the rules $r_i : r_i = \{\langle \Phi_i, \phi\rangle\}_{\phi \in S}$ $(i \in N)$, and assume that $R_M = \{r_i\}_{i \in N}$. Then the following theorem holds:

(1.6) $\langle R_M, 0\rangle$ is the M-weakest system with the Tarski's property.

Let us note as a comment to the theorem (1.6) that e.g. the system $\langle R_{Z_2}, \{p \to p\}\rangle$ weaker than the Tarski's system $\langle R_{0^*}, A\rangle$ has the Tarski's property. If we assume that in the M-weakest system (cf. [1], [5]) A' can't be an empty set, then $\{\langle R_M, \{\Phi\}\rangle\}_{\Phi \in M}$ is the set of all minimal systems which belongs to T_M, in other words $\langle R_M, \{\Phi\}\rangle$ $(\Phi \in M)$ is the weakest system with the Tarski's property in the class $\{\langle R, A\rangle \in T_M : \Phi \in Cn(R,A)\}$.

Let us assume moreover, that the relation $\approx$ of equivalence of systems $(\langle R, A\rangle \approx \langle R', A'\rangle)$ is defined as in [5]. The relation $\approx$ is an equivalence relation, thus we can define the quotient set $Q = \{\langle R, X\rangle : \langle R, X\rangle \in T_{Z_2}\}/\approx$ and the two following operations:

$$[\langle R_1, X_1\rangle]_\approx \overset{*}{\cap} [\langle R_2, X_2\rangle]_\approx = [\langle Der(R_1, X_1) \cap Der(R_2, X_2), Cn(R_1, X_1) \cap Cn(R_2, X_2)\rangle]_\approx$$

$$[\langle R_1, X_1\rangle]_\approx \overset{*}{\cup} [\langle R_2, X_2\rangle]_\approx = [\langle R_1 \cup R_2, X_1 \cup X_2\rangle]_\approx$$

Let us show that the following theorem holds:

(1.7) Let $[\langle R_1, X_1\rangle]_\approx \subseteq [\langle R_2, X_2\rangle]_\approx \equiv \langle R_1, X_1\rangle \preccurlyeq \langle R_2, X_2\rangle$, then $L = \langle Q, \subseteq, \overset{*}{\cap}, \overset{*}{\cup}\rangle$ is the lattice with 1_L and 0_L, where $1_L = [\langle R_{0^*}, A_2\rangle]_\approx$ (A_2 – the axioms of the Z_2) and $0_L = [\langle R_{Z_2}, 0\rangle]_\approx$.

2. Let $S = S^*$. Now we shall construct a system weaker than $\langle R_{0^*}, A\rangle$ which belongs to T_{Z_2}. Let

(2.1) $\Phi_1, \Phi_2, \Phi_3, \ldots$

be the sequence of all formulas from $S - Z_2$, and

(2.2) $A_i = \{e : e : At \to \{0, 1\}\ h^e(\Phi_i) = 0\}$ $(i = 1, 2, \ldots)$

Hence we have the family of sets $R = \{A\Phi_i\}_{i\in N}$.

Let us take only one element from every set belonging to R.

Hence we have the following sequence:

(2.3) $e\Phi_1, e\Phi_2, e\Phi_3, \ldots$

For every Φ_i from the sequence (2.1) we define the function $e^{\Phi_i} : At \to S$ $(i = 1, 2, 3, \ldots)$.

(2.4) $$e^{\Phi_i}(p_j) = \begin{cases} p \to p, & \text{if } \ e\Phi_i(p_j) = 1 \\ p, & \text{if } \ e\Phi_i(p_j) = 0 \end{cases} \qquad (j = 1, 2, \ldots)$$

and the following operation:

(2.5) $$\Phi_i^* = \begin{cases} \Phi_i, & \text{if } \ At(\{\Phi_i\}) = \{p\} \\ h^{e^{\Phi_i}}(\Phi_i) & \text{if } \ At(\{\Phi_i\}) \neq \{p\} \end{cases}$$

Using above definitions we can from the set

(2.6) $B = \{\phi \in S : \exists_{i\in N}\phi = \Phi_{i^*} \to p\}$

From (1.4) and (2.3) - (2.6) we get

(2.7) $\langle R_{o^*}, B\rangle \in T_{Z_2}$.

Since

(2.8) $Cn(R_{0^*}, B) \subsetneq Cn(R_{0^*}, A)$, then (cf. (2.7), (2.8)) the system $\langle R_{0^*}, B\rangle$ is weaker than $\langle R_{0^*}, A\rangle$. Note that every R_{0^*} – oversystem of the system $\langle R_{0^*}, B\rangle$ has the Tarski's property. Then, by means of (2.7) and (2.8) we get immediately the simple proof of the Tarski's theorem (1.1).

3. The are systems belonging to T_{Z_2} which have non-empty intersection with Tarski's system $\langle R_{0^*}, A\rangle$ and are not subsystems of $\langle R_{0^*}, A\rangle$. For example $\langle R_{0^*}, E(\mathfrak{M}_1)\rangle \in T_{Z_2}$ where $\mathfrak{M}_1 = \langle\{0,1,2,3\},\{1,2,3\},f^*\rangle$, and f^* is defined by the following table:

f^*	0	1	2	3
0	1	1	1	1
1	0	1	2	0
2	0	1	2	3
3	0	1	3	2

Observe that many matrix have the above property. Generally, if a matrix $\mathfrak{M}_n = \langle\{0,1,\ldots,n\},\{1,2,\ldots,n\},f^*\rangle$ fulfilled the following conditions:

1) if $x, y \in \{0,1\}$, then $f^*(x,y) = f_2^*(x,y)$
2) for any $\Phi \in Z_2$ such that $At(\{\Phi\}) = \{p\} : h^v(\Phi) \in \{2,3,\ldots,n\}$ for every $v : At \to \{2,3,\ldots,n\}$.

then the system $\langle R_{0^*}, E(\mathfrak{M}_n)\rangle \in T_{Z_2}$.

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