Title: Studies on Church's calculus

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Bulletin of the **S**ection of **L**ogic
Volume 5/2 (1976), pp. 57–61

reedition 2011 [original edition, pp. 57–62]

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STUDIES ON CHURCH'S CALCULUS*

An extended version of this abstract will appear in Reports on Mathematical Logic.

1. In Church's calculus we establish classes of equivalent formulas built from only one propositional variable p in order to obtain the theorem on existence exactly one Lindenbaum's extension for Church's system.

Moreover, we construct a class of finitely axiomatizable systems between Church's and Grzegorczyk's systems and we consider the problem of structural completeness of Church's calculus reduced to formulas formed from the variable p.

2. We use the following notations: S^C is the smallest set of well-formed formulas built by means of all propositional variables $At = \{p, q, r, p_1, p_2, \ldots\}$ and the connective of implication $\rightarrow$ ($\overline{\overline{S^C}} = \aleph^0$), $At(\Phi)$ denotes the set of all propositional variables occurring in Φ, Z_2 is the set of all two-valued tautologies from S^C. The notions of consequence operations $Sb(X)$ and $Cn(R_{0^*}, X)$, where $R_{0^*} = \{r_0, r_*\}$ (r_0 – the modus ponens rule, r_* – the substitution rule), are known (cf. [7], [6], pp. 105–106). $\mathcal{M}$ denotes the logical matrix, $E(\mathcal{M})$ is the set of all valid formulas in this matrix (from the language S^C).

The couple $\langle R_{0^*}, A_1\rangle$ with axioms $A_1 = \{p \rightarrow p, [p \rightarrow (p \rightarrow q)] \rightarrow (p \rightarrow q), [p \rightarrow (q \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)], (p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]\}$ and primitive rules $R_{0^*} = \{r_0, r_*\}$ is called Church's system (cf. [2]).

Two formulas Φ, $\Psi \in S^C$ are equivalent in Church's system, $\Phi \equiv \Psi$, iff $\Phi \rightarrow \Psi$, $\Psi \rightarrow \Phi \in Cn(R_{0^*}, A_1)$.

*An abstract this article is not to be reviewed.

$\langle R_{0^*}, A_2\rangle$ denotes Grzegorczyk's system with axioms $A_2 = A_1 \cup \{(p \to q) \to [p \to (p \to q)]\}$ (cf. [3], p. 102).

The set $L(Cn(R_{0^*}, X))$ of all Lindenbaum's extensions of the set $Cn(R_{0^*}, X)$ will be defined as follows: $L(Cn(R_{0^*}, X)) = \{Y \subseteq S^C : Cn(R_{0^*}, X) \subseteq Y = Cn(R_{0^*}, Y) \neq S^C \wedge \forall_{\Phi \in S^C \setminus Y} Cn(R_{0^*}, Y) \cup \{\Phi\}) = S^C\}$.

$SCpl$ is the set of all structural complete systems (cf. [5]).

3.

LEMMA 3.1.

$$\begin{aligned} &\forall\{At(\Phi) = \{p\} \Rightarrow \\ &\quad ([\Phi \in Z_2 \Rightarrow \Phi \equiv p \to p \vee \Phi \equiv p \to (p \to p) \\ &\quad\quad \vee \Phi \equiv [p \to (p \to p)] \to (p \to p)] \\ &\quad \wedge [\Phi \notin Z_2 \Rightarrow \Phi \equiv p \vee \Phi \equiv [p \to (p \to p)] \to p \\ &\quad\quad \vee \Phi \equiv \{[p \to (p \to p)] \to (p \to p)\} \to p])\} \end{aligned}$$

THEOREM 3.2. $\forall_{\Phi \in S^C}\{At(\Phi) = \{p\} \Rightarrow [\Phi \in E(\mathcal{M}) \Leftrightarrow \Phi \in Cn(R_{0^*}, A_1)]\}$, *where* $\mathcal{M} = \langle\{0,1,2,3\}, \{1,2,3\}, f^{\to}\rangle$ *and*

$f^{\to}$	0	1	2	3
0	1	1	1	1
1	0	1	0	0
2	0	1	2	3
3	0	1	0	2

Lemma 3.1 and Theorem 3.2 allow us to formulate and algorithm concerned equivalents formulas.

COROLLARY 3.3.

$$\begin{aligned} &\forall_{e:At \to \{0,1,2,3\}} \forall_{\Phi \in S^C}\{At(\Phi) = \{p\} \wedge e(p) = 3 \Rightarrow \\ &\quad \{\Phi \in Z_2 \Rightarrow ([h^e(\Phi) = 0 \Rightarrow \Phi \equiv p \to (p \to p)] \\ &\quad\quad \wedge [h^e(\Phi) = 1 \Rightarrow \Phi \equiv [p \to (p \to p)] \to (p \to p)] \\ &\quad\quad \wedge [h^e(\Phi) = 2 \Rightarrow \Phi \equiv p \to p])\} \\ &\quad \wedge\{\Phi \notin Z_2 \Rightarrow ([h^e(\Phi) = 0 \Rightarrow \Phi \equiv \{p \to (p \to p)] \to (p \to p)) \to p] \\ &\quad\quad \wedge [h^e(\Phi) = 1 \Rightarrow \Phi \equiv [p \to (p \to p)] \to p \\ &\quad\quad \wedge [h^e(\Phi) = 3 \Rightarrow \Phi \equiv p])\})\} \end{aligned}$$

We say that propositional calculus $\langle R_{0^*}, X\rangle$ has $\mathcal{T}$ property iff the set of all two-valued implicational tautologies is the only one Lindenbaum's extension for $\langle R_{0^*}, X\rangle$.

THEOREM 3.4. $\langle R_{0^*}, A_1\rangle \in \mathcal{T}$.

4. There exist denumerable many finitely axiomatizable systems between Church's and Grzegorczyk's calculus.

Let $\underline{R} = \{\langle R_{0^*}, A_1 \cup \{\alpha_n\}\rangle\}_{n\in N}$, where $\alpha_n = \underbrace{p \to (p \to \ldots (p}_{n-1 \text{ times}} \to q)\ldots) \to [\underbrace{p \to (p \to \ldots (p}_{n \text{ times}} \to q)\ldots)]$, and $\mathcal{M}_n = \langle\{0,1,2,\ldots,n{+}1\}, \{1,2,\ldots, n+1\}, f_n^{\to}\rangle$ $n = 1,2,\ldots$ where

$f_n^{\to}$	0	1	2	3	...	$n-1$	n	$n+1$
0	1	1	1	1	...	1	1	1
1	0	1	0	0	...	0	0	0
2	0	1	2	3	...	$n-1$	n	$n+1$
3	0	1	0	2	...	$n-2$	$n-1$	n
.	.	.	.			.	.	.
.	.	.	.			.	.	.
.	.	.	.			.	.	.
$n-1$	0	1	0	0	...	2	3	4
n	0	1	0	0	...	0	2	3
$n+1$	0	1	0	0	...	0	0	2

Let us notice that $\alpha_n \in E(\mathcal{M}_n)$ since for $v(p) = 3$ and $v(q) = n+1$ $h^v(\alpha_n) = 0$, and $\alpha_{n+1} \in E(\mathcal{M}_n)$.

THEOREM 4.1.

a) $Cn(R_{0^*}, A_1) \subseteq \bigcap_{n\in N} E(\mathcal{M}_n)$

b) $E(\mathcal{M}_m) \subsetneq E(\mathcal{M}_n)$ *for* $m > n$*, where* $m, n \in N$

c) $Cn(R_{0^*}, A_1 \cup \{\alpha_{n+1}\}) \subseteq E(\mathcal{M}_n) \wedge Cn(R_{0^*}, A_1 \cup \{\alpha_{n+1}) \subsetneq E(\mathcal{M}_{n+1})$ *for every* $n \in N$.

5. S_p^C is the smallest set satisfying the conditions:

a. $p \in S_p^C$

b. $\varphi, \psi \in S_p^C \Rightarrow \varphi \to \psi \in S_p^C$.

THEOREM 5.1.

a) $\langle R_{0^*}, A_p\rangle \in SCpl$

b) $\langle R, Sb(A_p) >\notin SCpl$, *where* $A_p = Cn(R_{0^*}, A_1) \cap S_p^C$.

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