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ON LINDENBAUM'S EXTENSIONS*
(Part C)

The full text with detailed proofs will appear in Reports on Mathematical Logic.

1. The purpose of the present paper is to discuss the theorems on existence one Lindenbaum's extensions for finitely axiomatizable systems. This problem is connected with the well known Tarski's theorems (cf. [15]) and the paper [2] devoted not finitely axiomatizable systems which have only one Lindenbaum's extension. Moreover among other things we will consider systems with different than R_{0^*}-rules and the system with $\mathcal{T}_M$-property for $M \neq Z_2^i$ ($i = 1, 2, 3$ cf. [3]).

2. Introduction. By S^i ($i = 1, 2, 3$) we denote the smallest sets of well-formed formulas ($\overline{\overline{S^i}} = \aleph_0$) built by means of all propositional variables $At = \{p, q, r, p_1, p_2, \ldots\}$ and connectives: ($\rightarrow$) implication; ($\rightarrow$) implication, ($\sim$) negation, (.) conjunction and (+) disjunction; ($\sim$) negation, (.) conjunction, (M) possibility, respectively. $At(X)$ ($X \subseteq S^i$) denotes the set of all propositional variables occurring in Φ, for every $\Phi \in X$. R is a set of rules of inference. R_{0^*} denotes the set $\{r_0, r_*\}$ (r_0 – the modus ponens rule, r_* – the substitution rule). $\langle R, X\rangle \in Cns$ means that $Cn(R, X) \neq S^i$ ($i = 1, 2, 3$). $R(X)$ means that the set X is closed with respect to the rules belonging to the set R. Let $\mathcal{M}$ be the logical matrix, then $S(\mathcal{M})$ is the set of all valid formulas in this matrix. We define now the general notion of $\mathcal{T}_M$-property (for any $M \not\subseteq S^i$, $i = 1, 2, 3$): the system $\langle R, X\rangle$ has $\mathcal{T}_M$-property iff M is the sole Lindenbaum's extension. For simplicity the symbol $\mathcal{T}_i$ instead of $\mathcal{T}_{Z_2^i}$ will be used where Z_2^i is the set of all two-valued

*As abstract this article is not to be reviewed.

tautologies ($i = 1, 2, 3$). Axioms of Tarski's systems $\langle R_{0^*}, A_i\rangle$ ($A_i \subseteq S^i$, $i = 1, 2$) are as follows (cf. [15]):

$A_1 = \{p \to (q \to p),\ p \to [(p \to q) \to q],\ (q \to s) \to [(p \to q) \to (p \to s)]\ \}$

$A_2 = \{p \to \sim\sim p,\ q \to (p \to q),\ \sim p \to (p \to q), p \to [\sim q \to \sim (p \to q)], p \to (p + q),\ q \to (p + q),\ \sim p \to [\sim q \to \sim (p + q)],\ p \to [q \to (p \cdot q)], \sim p \to \sim (p \cdot q),\ \sim q \to \sim (p \cdot q)\}$

For S^i we denote $S.9 = Cn(R_1, A_3)$ and $S.8 = Cn(R_1, A_4)$ where A_3 and A_4 are the sets of axioms of well-known modal systems (cf. [9],[14],[8]). $R_1 = \{r_0, r_*, r_a, r_E\}$, where r_o is the modus ponens rule, r_* is the subsitution rule, r_a is defined by the scheme $\Phi,\ \Psi/\Phi \cdot \Psi$, r_E is defined by the scheme $\alpha(\Phi), \Phi = \Psi/\alpha(\Psi)$. A_5 is the set of axioms of Church's system (cf.[6],[4]). Let $\#$ be the relation between two systems: $\langle R_{o*}, X\rangle\#\langle R_{o*}, X'\rangle$ iff the sets $Cn(R_{o*}, X) \cap Cn(R_{o*}, X')$, $Cn(R_{o*}, X) - Cn(R_{o*}, Y)$, $Cn(R_{o*}, Y) - Cn(R_{o*}, X)$ are non empty. Cpl is the class of Post-complete systems (cf. [10]). $L^{\phi}(Cn(R, X))$ is the set of all Lindenbaum-Asser extensions of the set $Cn(R, X)$ for $\phi \notin Cn(R, X)$ (cf. [1], [11], [5]).

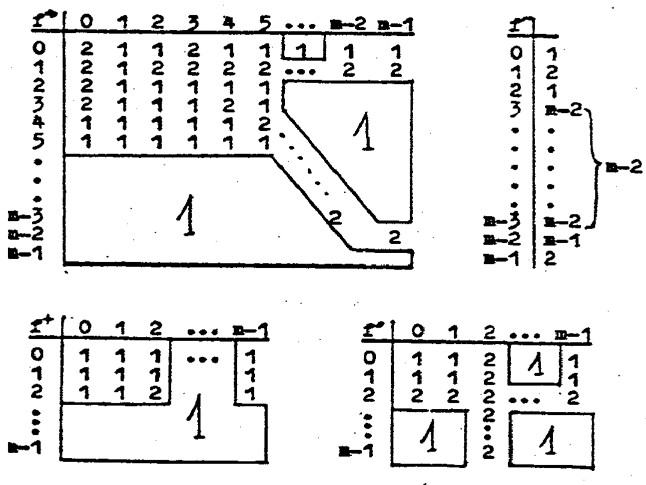
3. We consider for S^2 the following set of axioms (cf. [3]);

$X_1 = \{\Phi_i \to\sim\sim \Phi_i, \Phi_i \to [\sim \Phi_j \to\sim (\Phi_i \to \Phi_j)], \Phi_i \to (\Phi_j \to \Phi_i), \sim \Phi_i \to (\Phi_i \to \Phi_j), \Phi_i \to (\Phi_i + \Phi_j), \Phi_j \to (\Phi_i + \Phi_j), \sim \Phi_i \to [\sim \Phi_j \to\sim (\Phi_i + \Phi_j)], \Phi_i \to [\Phi_j \to (\Phi_i + \Phi_j)], \sim \Phi_i \to\sim (\Phi_i \cdot \Phi_j), \sim \Phi \to (\Phi_i + \Phi_j)$: $i \in \{1, 2, 3, 4\}$, $j \in \{5, 6, 7, 8\}$ where $\Phi_1 = p \to q$, $\Phi_2 = p.q$, $\Phi_3 = p + q$, $\Phi_4 =\sim p$, $\Phi_5 = r \to s$, $\Phi_6 = r.s$, $\Phi_7 = r + s$, $\Phi_8 =\sim r$.

Gödel's calculi G_n are given by the matrices $\mathcal{M}_n$ of Gödel. The sequence of matrices $\mathcal{M}_n$ was introduced by Gödel in [7] and was axiomatized by Thomas in [16]. Anderson proved that $G_n = Cn(R_{0^*}, H \cup \{T_n\})$, where H is the set of axioms of the intuitionistic logic and T_n is defined as follows:

$$T_n = p_1 + (p_1 \to p_2) + (p_2 \to p_3) + \ldots + (p_{n-1} \to p_n) + \sim p_n \quad (n \geqslant 4).$$

Using the matrix $\mathcal{N}_1^{(n-1)} = \langle\{0, 1, \ldots, m-1\}, \{1\}, f^{\to}, f^{\sim}, f^*, f^+\rangle$ ($m \geqslant 6$) where



we can prove that $T_{m-1} \notin Cn(R_{0^*}, X_1 \cup \{T_4, T_5, \ldots, T_{m-2}\})$ since $Cn(R_{0^*}, X_1 \cup \{T_4, T_5, \ldots, T_{m-2}\}) \subseteq E(\mathcal{N}_1^{(m-1)})$ and for $v(p_1) = 2$, $v(p_2) = 0$, $v(p_3) = 3$, $\ldots$, $v(p_{m-2}) = m-2$, $v(p_m) = m-1$ $(v : At \to |\mathcal{N}_1^{(m-1)}|)$ we have $h^v(T_{m-1}) = 2$. Moreover, for every $m \geqslant 6$ it follows that $p \to (q \to p) \in Cn(R_{0^*}, A_2) - Cn(R_{0^*}, X_1 \cup (T_4, T_5, \ldots, T_{m-2}\})$. This can be shown by means of the matrix

$\mathcal{M}_1 = \langle \{0,1,2\}, \{1\}, f^{\to}, f^{\sim}, f^{*}, f^{+} \rangle$ where

$f^{\to}$	0	1	2
0	2	1	2
1	2	1	2
2	1	1	1

$f^{\sim}$	
0	2
1	2
2	1

f^{*}	0	1	2
0	1	1	2
1	1	1	2
2	2	2	2

f^{+}	0	1	2
0	1	1	1
1	1	1	1
2	1	1	2

Hence the following lemma holds:

LEMMA 3.1. *For every* $m, n \geqslant 4$ $(m \neq n)$

a. $\langle R_{0^*}, X_1 \cup \{T_n\}\rangle \# \langle R_{0^*}, X\rangle$

b. $\langle R_{0^*}, X_1 \cup \{T_n\}\rangle \# \langle R_{0^*}, X_1 \cup \{T_m\}\rangle$

c. $\langle R_{0^*}, X_1 \cup \{T_n\}\rangle \# \langle R_{0^*}, A_2\rangle$

d. $\langle R_{0^*}, X_1 \cup \{T_n\}\rangle \in \mathcal{T}_2$

THEOREM 3.2.

a. $\{\langle R_{0^*}, X_1 \cup \{T_n\}\rangle\}_{n\geqslant 4}$ *is the family of finitely axiomatizable systems "on the edge" of Tarski's system* $\langle R_{0^*}, A_2\rangle$.

b. $\{\langle R_{0^*}, X_1 \cup \bigcup_{k=1}^{n} \{T_k\}\rangle\}_{n\geqslant 4}$ *is the family of the ascending systems with* $\mathcal{T}_2$*-property satisfying the following two conditions:*

1. $\langle R_{0^*}, X_1 \cup \bigcup_{k=1}^{n} \{T_k\}\rangle \# \langle R_{0^*}, H\rangle$
2. $\langle R_{0^*}, X_1 \cup \bigcup_{k=1}^{n} \{T_k\}\rangle \# \langle R_{0^*}, A_2\rangle$

We introduce the sequence $\alpha_1 = 2, \ldots, \alpha_{n+1} = \alpha_n + (n+2)$ and $\Gamma_0 = 0, \ldots, \Gamma_{n+1} + 1$. Let us put $p = p_{\Gamma_0+2}$, $q = p_{\Gamma_0+1}$ and define the functions $e_{i_k} : At \to S^2$ $(i = \{1,2,3,4\}, k \in N - \{1\})$ as follows:

$$\begin{array}{lll} e_{i_2}(p_{\Gamma_0+1} = \delta_i(p_{\Gamma_1+1}, p_{\Gamma_1+2}) & & \\ e_{i_2}(p_{\Gamma_0+2} = \delta_i(p_{\Gamma_1+3}, p_{\Gamma_1+4}) & & \delta_1(\Phi,\Psi) = \Phi \to \Psi \\ \ldots\ldots\ldots\ldots\ldots\ldots & i \in \{1,2,3\} & \delta_2(\Phi,\Psi) = \Phi \cdot \Psi \\ e_{i_{n+1}}(p_{\Gamma_{n-1}+1} = \delta_i(p_{\Gamma_n+1}, p_{\Gamma_n+2}) & & \delta_3(\Phi,\Psi) = \Phi + \Psi \\ \ldots\ldots\ldots\ldots\ldots\ldots & & \end{array}$$

$$e_{i_{n+1}}(p_{\Gamma_{n-1}+2^n} = \delta_i(p_{\Gamma_n+(2^{n+1}-1)}, p_{\Gamma_n+2^{n+1}})$$

and

$$\begin{array}{l} e_{4_2}(p_{\Gamma_0+1}) = \delta_4(p_{\Gamma_1+1}) \\ e_{4_2}(p_{\Gamma_0+2}) = \delta_4(p_{\Gamma_1+3}) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ e_{4_{n+1}}(p_{\Gamma_{n-1}+1}) = \delta_4(p_{\Gamma_n+1}) \quad \text{where} \quad \delta_4(\Phi) = \sim \Phi \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

$$e_{4_{n+1}}(p_{\Gamma_{n-1}+2^n}) = \delta_4(p_{\Gamma_n+(2^{n+1}-1)})$$

Hence we obtain the sets of formulas

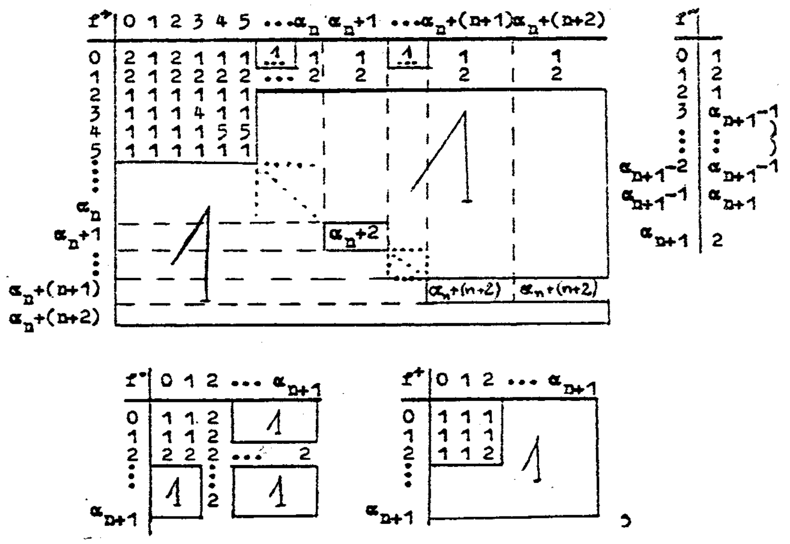
$$X_2 = \{\Phi \in S^2 : \exists_{\lambda\in A_2}\exists_{e:At\to S^2}\exists_{i,k\in\{1,2,3,4\}}[\Phi = h^e(\lambda) \wedge e(p_{\Gamma_0+1}) = \\ = e_{i_2}(p_{\Gamma_0+1}) \wedge e(p_{\Gamma_0+2}) = e_{k_2}(p_{\Gamma_0+2})]\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$X_{n+1} = \{\Phi \in S^2 : \exists_{\lambda\in X_n}\exists_{e:At\to S^2}\exists_{\{k(1),\ldots,k(2^n)\}\subseteq\{1,2,3,4\}}[\Phi = h^e(\lambda)\wedge \\ \wedge e(p_{\Gamma_{n-1}+1}) = e_{k(1)_{n+1}}(p_{\Gamma_{n-1}+1} \wedge \ldots \wedge e(p_{\Gamma_{n-1}+2^n}) = \\ = e_{k(2^n)_{n+1}}(p_{\Gamma_{n-1}+2^n})]\}$$

By using the matrix

$\mathcal{N}_n^{(n+1)} = \langle\{0, 1, \ldots, \alpha_{n+1}\}, \{1\}, f^{\rightarrow}, f^{\sim}, f^{*}, f^{+}\rangle$, where



we can prove that $Cn(R_{0^*}, C_{n+2}) \subseteq E(\mathcal{N}_2^{(n+1)})$. At the same time $\Phi_{n+1} = h^{e_{1_{n+1}}}(h^{e_{1_n}}(\ldots(h^{e_{1_2}}(q \to (p \to q)))\ldots)) \in Cn(R_{0^*}, X_{n+1}) - Cn(R_{0^*}, X_{n+2})$ for $v(p_{\Gamma_n+1}) = v(p_{\Gamma_n+2}) = \ldots = v(p_{\Gamma_n+2^{n+1}}) = \alpha_n + 1$ fulfills the equality $h^v(\Phi_{n+1}) = \alpha_{n+1}$ and hence $\Phi_{n+1} \notin Cn(R_{0^*}, X_{n+2})$.

LEMMA 3.3. *For every* $n \geqslant 2$

a. $\langle R_{0^*}, X_n\rangle \in \mathcal{T}_2$

b. $Cn(R_{0^*}, X_n) \nsubseteq Cn(R_{0^*}, X_{n+1})$

c. $Cn(R_{0^*}, X_n) \subset Cn(R_{0^*}, A_2)$

THEOREM 3.4. *There exists a family of the descending finitely axiomatizable systems which are subsystems of Tarski's system* $\langle R_{0^*}, A_2\rangle$ *of the power of* $\aleph_0$.

This result can be obtained for S^1 and $\langle R_{0^*}, A_1\rangle$.

THEOREM 3.5.

a. $\forall_{M \subsetneq S^2} \forall_{X \supseteq Z_2^2} [Cn(\{r_0\}, X) = M \vee \langle\{r_0\}, X\rangle \notin \mathcal{T}_M]$

b. $\forall_{M \not\subseteq S^2}[\langle\{r_0\}, X\rangle \in \mathcal{T}_M \wedge Z_2^2 \subseteq X \Rightarrow \langle\{r_0\}, X\rangle \in Prime]$
where $Prime$ *is the class of well-known prime systems.*
Because $\forall_{\phi \in Z_2^2 - H}\forall_{Y \in L^\phi(Cn(\{r_0\},H))} Y \notin Cpl$ *and*
$\forall_{Z_2^2 \subseteq X}\forall_{\phi \notin Cn(\{r_0\},X)}[Y_1, \ldots, Y_n \in L^\phi(Cn(\{r_0\}, X)) \Rightarrow L^\phi(Cn(\{r_0\}, \bigcap_{i=1}^{n} Y_i)) = \{Y_1, \ldots, Y_n\}]$,
($L^\phi(Cn(\{r_0\}, X))$ *can be exchanged by* $L(Cn(\{r_0\}, X))$ *and* $L^\phi(CN(\{r_0\}, \bigcap_{i=1}^{n} Y_i))$ *by* $L(Cn(\{r_0\}, \bigcap_{i=1}^{n} Y_i)))$*; so we introduce the notion* $\mathcal{T}_{M_\phi}$ *as follows* $(M \not\subseteq S^2)$*:*
$\langle R, X\rangle \in \mathcal{T}_{M_\phi} \Leftrightarrow \forall_{Y \in L^\phi(Cn(R,X))} Y = M$

4. For S^3 the following theorem can be proved (this theorem was announced in [12] without proof):

THEOREM 4.1. $\langle R_1, A_4\rangle \in \mathcal{T}_{S.9}$

PROOF. By induction on the length of formula we obtain the following lemma:

1. $\forall_{\Phi \in S^3}\{\exists_{e: At \to \{(p\to p), \sim(p \to p)\}}\ \exists_{\Phi \in S^3} h^e(\Phi) = \Phi \Rightarrow$
$([\Phi \in S.9 \Rightarrow (\Phi \overline{\overline{S.8}} \sim (p \to p) \vee \Phi \overline{\overline{S.8}} \sim M(p \to p))] \wedge$
$[\Phi \in S.9 \Rightarrow (\Phi \overline{\overline{S.8}} (p \to p) \vee \Phi \overline{\overline{S.8}} M(p \to p))]\}$
Let $\mathcal{M}_{S.9} = \langle\{1, 2, 3, 4\}, \{1, 2\}, f^{\to}, f^{\vee}, f^{*}, M, L\rangle$ where (cf. [9])

$f^{\to}$	1	2	3	4
1	2	4	4	4
2	2	2	4	4
3	2	4	2	4
4	2	2	2	2

$f^{\sim}$	
1	4
2	3
3	2
4	1

f^{*}	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

p	Mp	Lp
1	1	2
2	1	4
3	1	4
4	3	4

We have (cf. [12], [13]) that $\langle R_1, A_3\rangle \in Cpl$, hence it follows that $S.9 = E(\mathcal{M}_{S.9})$. Let us take $\Phi \notin Cn(R_1, A_3)$ hence $\exists_{v: At \to |\mathcal{M}_{S.9}|} h^v(\Phi) \notin \{1, 2\}$
Let us define $e_1 : At \to S^3$ as follows:

$$e_1(p_j) = \begin{cases} p \to p & \text{if } \ w(p_j) = 2 \\ M(p \to p) & \text{if } \ v(p_j) = 1 \\ \sim (p \to p) & \text{if } \ v(p_j) = 3 \\ \sim M(p \to p) & \text{if } \ v(p_j) = 4 \end{cases}$$

2. $h^{e_1}(\Phi) \in Cn(R_1, A_3)$. On the ground of 1 we should consider two cases:
   a. $h^{e_1}(\Phi)_{\overline{\overline{S.8}}} \sim (p \to p)$
   b. $h^{e_1}(\Phi)_{\overline{\overline{S.8}}} \sim (M(p \to p))$

   a. $h^{e_1}(\Phi), h^{e_1}(\Phi) \to\sim (p \to p) \in Cn(R_1, A_3 \cup \{\Phi\})$, hence $\sim (p \to p) \in Cn(R_1, A_3 \cup \{\Phi\})$. $p\cdot \sim p \to q$, $p \to p \in Cn(R_1, A_3)$ and, using the rule r_a, we have that $(p \to p)\cdot \sim (p \to p) \in Cn(R_1, A_3 \cup \{\Phi\})$. Consequently, $Cn(R_1, A_3 \cup \{\Phi\}) = S^3$.

   b. $h^{e_1}(\Phi), h^{e_1}(\Phi) \to\sim M(p \to p) \in Cn(R_1, A_3 \cup \{\Phi\})$ hence $\sim M(p \to p) \in Cn(R_1, A_3 \cup \{\Phi\})$. Since $M(p \to p) \in Cn(R_1, A_3)$ then by the rule r_a $M(p \to p)\cdot \sim M(p \to p) \in Cn(R_1, A_3 \cup \{\Phi\})$ and by $p\cdot \sim p \to q$ we have that $Cn(R_1, A_3 \cup \{\Phi\} = S^3$.

   Hence $\langle R_1, A_4 \rangle \in \mathcal{T}_{S.9}$.

   Some examples of the systems for S^1 with $\mathcal{T}_1$-property with different than R_{0^*}-primitive rules. We consider the following sets of axioms:
   $A_6 = \{[(p \to p) \to p] \rightleftarrows p,\ [p \to (p \to p)] \rightleftarrows p \to p),$
   $[(p \to p) \to (p \to p)] \rightleftarrows (p \to p)\},$
   $A_7 = \{(p \to q) \to [(q \to s) \to (p \to s)],\ (q \to s) \to [(p \to q) \to (p \to s)]\} \cup A_6$ and primitive rules $R_2 = \{r_0, r_*, r_E\}$, where r_E is the rule of the scheme $\Phi \to \Psi$, $\Psi \to \Phi$, $\Theta(\Phi)/\Theta(\Psi)$.

THEOREM 4.2.

a. $\langle R_2, A_6 \rangle \in \mathcal{T}_1$
b. $\langle R_2, A_7 \rangle \in \mathcal{T}_1 \wedge Cn(R_2, A_7) \subset Cn(R_{0^*}, A_1) \wedge r_E \in Der(R_{0^*}, A_7)$
c. There exists finitely axiomatizable system $\langle R_2, A_8 \rangle$ *with* $\mathcal{T}_1$*-property which is a subsystem of Church's system and* $At(A_8) = \{p\}$.

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