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A NEW PROOF OF STRUCTURAL COMPLETENESS OF LUKASIEWICZ'S LOGICS

The problem of structural completeness of the finite-valued Łukasiewicz's sentential calculi was investigated and solved in [4], [7], [6]. The present paper contains a new proof of all these results.

I. Let $\langle S; F_1, \dots, F_n \rangle$ be a propositional language. A matrix $M = \langle |M|, |M|^*; f_1, \dots, f_n \rangle$ of this language is embeddable in a matrix $N = \langle |N|, |N|^*; g_1, \dots, g_n \rangle$ ($M \subseteq N$) iff there exists a monomorphism $h : M \rightarrow N$. The symbol $N \times M$ stands for the product of matrices and the structural consequence generated by M (cf. [1]) is denoted by \vec{M} . We know (cf. [9]) for every M, N :

$$(1.1) \quad N \subseteq M \Rightarrow \vec{M} \leq \vec{N}.$$

$$(1.2) \quad \vec{M} = \overline{M/\sim} \text{ for every congruence } \sim \text{ of matrix } M.$$

Moreover, one can prove (cf. [11]) that

$$(1.3) \quad \overline{M \times N}(X) = \begin{cases} \vec{M}(X) \cap \vec{N}(X) & \text{if } X \in \text{Sat}(M) \text{ and } X \in \text{Sat}(N) \\ S & \text{if } X \notin \text{Sat}(M) \text{ or } X \notin \text{Sat}(N) \end{cases}$$

We recall that $X \in \text{Sat}(M)$ iff there exists $v : At \rightarrow |M|$ such that $h^v(X) \subseteq |M|^*$. A consequence Cn (or a propositional calculus $\langle R, A \rangle$) is structurally complete iff every structural and permissible rule of this consequence (of this calculus) is at the same time a derivable rule of Cn (of $\langle R, A \rangle$). Let $L_{Cn(0)}$ denotes the Lindenbaum's matrix $\langle S, Cn(O); F_1, \dots, F_n \rangle$ of a consequence Cn .

The following theorem characterize a structurally complete consequence in the set of all structural consequences:

- (1.4) $Cn \in Struct \Rightarrow [Cn \in SCpl \Leftrightarrow Cn = \vec{L}_{Cn(0)}]$
 (1.5) $Cn \in Struct \Rightarrow \{Cn \in SCpl \Leftrightarrow \forall Cn_1 \in Struct [Cn_1(0) = Cn(0) \Rightarrow Cn_1 \leq Cn]\}$

Theorem (1.4) is proved in [5] and (1.5) can be found in [2]. Let Sb be the consequence operation based on the substitution rule only. It is easy to see that:

$$(1.6) \quad Cn \in Struct \cap SCpl \Rightarrow CnSb \in SCpl.$$

From the finitistic point of view by a rule of inference we mean an operation with finite set of premises. Then we must consider finitistic structural completeness. For finitistic structural completeness (to be denoted by $SCpl_F$) it can be proved (cf. [2]) that:

$$(1.7) \quad Cn \in Struct \Rightarrow \{Cn \in SCpl_F \Leftrightarrow \forall Cn_1 \in Struct \cap Fin [Cn_1(O) \Rightarrow Cn_1 \leq Cn]\}$$

II. In the sequel, the symbol S will denote the set of formulas built up by means of propositional variables and the following connectives (notation follows that of Łukasiewicz): C (implication), K (conjunction), E (equivalence), A (disjunction) and N (negation). The symbol S^p stands for the set of all positive formulas. All results from I will be applied to these language. Let $M_n = \langle \{0, \frac{1}{n-1}, \dots, 1\}, \{1\}; c, a, k, e, n \rangle$ be the n -valued Łukasiewicz's matrix and let $M_n^p = \langle |M_n|, \{1\}; c, a, k, e \rangle$ be a positive reduct of this matrix. Moreover, we put $R_{0*} = \{r_0, r_*\}$ and $R_0 = \{r_0\}$, where r_0 is the modus ponens rule and r_* is the substitution rule. It is known (cf. [8]) that for $\langle R_{0*}, A_n \rangle$ (the n -valued Łukasiewicz's calculus) and for $\langle R_{0*}, A_n^p \rangle$ (the positive n -valued Łukasiewicz's calculus) we have:

$$(2.1) \quad Cn(R_0, Sb(A_n) \cup X) = \vec{M}_n(X) \quad \text{for every } X \subseteq S.
 Cn(R_0, Sb(A_n^p) \cup X) = \vec{M}_n^p(X) \quad \text{for every } X \subseteq S^p.$$

From now onward, the symbol L_n will denote the Lindenbaum's matrix of $\langle R_{0*}, A_n \rangle$ and L_n^p will be the Lindenbaum's matrix of $\langle R_{0*}, A_n^p \rangle$. Let us define a relation \approx_n on S .

$$(2.2) \quad \alpha \approx_n \beta \Leftrightarrow E\alpha\beta \in E(M_n).$$

From this definition it directly follows that $\alpha \approx_n \beta$ iff $h^v\alpha = h^v\beta$ for every

$v : At \rightarrow |M_n|$. The relation \approx_n is a congruence of the Lindenbaum's matrix L_n . By induction one can prove that for every $v : At \rightarrow |M_n|$ and for every formula $C^k pq$ ($C^{k+1} pq = CpC^k pq$)

$$(2.3) \quad h^v(C^k pq) = \min\{1, k(1 - vp) + vq\}$$

Hence we obtain that for every $v : At \rightarrow |M_n|$

$$(2.4) \quad h^v(C^{n-1} pq) = \begin{cases} 1 & \text{if } vp \neq 1 \\ vq & \text{if } vp = 1 \end{cases}$$

$$h^v(C^{n-2} pq) = \begin{cases} vq & \text{if } vp = 1 \\ vp & \text{if } vp = \frac{n-2}{n-1} \text{ and } vq = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Since in every M_n we have $h^v(CCpq) = \max\{vp, vq\} = a(vp, vq)$ we infer from (2.4) that:

$$(2.5) \quad h^v(CCC^{n-2} pqp) = \begin{cases} \frac{n-2}{n-1} & \text{if } vp = \frac{n-2}{n-1} \text{ and } vq = 0 \\ 1 & \text{if } vp \neq \frac{n-2}{n-1} \text{ or } vq \neq 0 \end{cases}$$

$$h^v(CC^{n-1} C_{n-2} pqqCC^{n-1} pqq) = \begin{cases} 0 & \text{if } vp = \frac{n-2}{n-1} \text{ and } vq = 0 \\ 1 & \text{if } vp \neq \frac{n-2}{n-1} \text{ or } vq \neq 0 \end{cases}$$

III. We shall prove now the following fundamental theorem on structural completeness of Łukasiewicz's logics.

$$(3.1) \quad \overrightarrow{M_n \times M_2} \in SCpl \text{ for every } n \geq 2.$$

PROOF. Suppose that $n \geq 2$ is a fixed natural number. For every $x \in |M_n|$ let α_x^0, α_x^1 be two formulas defined as follows:

$$\alpha_1^1 = Cpp, \quad \alpha_{\frac{n-2}{n-1}}^1 = CCC^{n-2} pqp,$$

$$\alpha_0^1 = CC^{n-1} C^{n-2} pqqCC^{n-1} pqq \text{ (if } n > 2),$$

$$\alpha_x^1 = C^{x(n-1)} \alpha_{\frac{n-2}{n-1}}^1 \alpha_0^1 \text{ for } 0 \leq x < \frac{n-2}{n-1}, \quad \alpha_x^0 = N\alpha_{1-x}^1.$$

From (2.3) and (2.5) it follows that for every $v : At \rightarrow |M_n|$

$$h^v(\alpha_x^i) = \begin{cases} i & \text{if } vp \neq \frac{n-2}{n-1} \text{ or } vq \neq 0 \\ x & \text{if } vp = \frac{n-2}{n-1} \text{ and } vq = 0 \end{cases}$$

It is easy to verify that for every $v : At \rightarrow |M_n|$

$$h^v(C\alpha_x^i\alpha_y^j) = h^v(\alpha_{c(x,y)}^{c(i,j)}), \quad h^v(A\alpha_x^i\alpha_y^j) = h^v(\alpha_{a(x,y)}^{a(i,j)}),$$

$$h^v(K\alpha_x^i\alpha_y^j) = h^v(\alpha_{k(x,y)}^{k(i,j)}), \quad h^v(E\alpha_x^i\alpha_y^j) = h^v(\alpha_{e(x,y)}^{e(i,j)}),$$

$$h^v(N\alpha_x^i) = h^v(\alpha_{n(x)}^{n(i)}).$$

$$\text{Hence, } C\alpha_x^i\alpha_y^j \approx_n \alpha_{c(x,y)}^{c(i,j)}, \quad A\alpha_x^i\alpha_y^j \approx_n \alpha_{a(x,y)}^{a(i,j)},$$

$$K\alpha_x^i\alpha_y^j \approx_n \alpha_{k(x,y)}^{k(i,j)}, \quad E\alpha_x^i\alpha_y^j \approx_n \alpha_{e(x,y)}^{e(i,j)}, \quad N\alpha_x^i \approx_n \alpha_{n(x)}^{n(i)}.$$

From the above it follows that a mapping $f : |M_n| \times |M_2| \rightarrow |L_n / \approx_n|$ defined as follows $f(\langle x, i \rangle) = [\alpha_x^i]$ is a monomorphism. Hence, $M_n \times M_2 \subseteq L_n / \approx_n$ and from (2.1) and (2.2) we obtain $\overrightarrow{M_n \times M_2} \supseteq \overrightarrow{L_n / \approx_n} = \overrightarrow{L_n}$. On the other hand, since $\overrightarrow{M_n}(0) = \overrightarrow{L_n}(0) = \overrightarrow{M_2 \times M_n}(0)$ we infer from (1.4) and (1.5) that $\overrightarrow{M_n \times M_2} \subseteq \overrightarrow{L_n}$. Hence $\overrightarrow{M_n \times M_2} = \overrightarrow{L_n} \in SCpl$.

Note that this theorem is proved without McNaughton's criterion and the representation theorem for Łukasiewicz's algebras. Theorem (3.1) can be immediately deduced from results of [3] and [10]. From (1.3) it follows that

$$\overrightarrow{M_n \times M_2}(X) = \begin{cases} \overrightarrow{M_n}(X) & \text{if } X \in Sat(M_2) \\ S & \text{if } X \notin Sat(M_2). \end{cases}$$

It is easy to see that $\overrightarrow{M_n \times M_2} \neq \overrightarrow{M_n}$ (for example, if $\alpha = C^{n-1}CCNpppNCCNppp$, then $\overrightarrow{M_n}(\alpha) \neq S$ and $\alpha \notin Sat(M_2)$) for $n > 2$. From this and (1.7) it follows that (cf. [7]):

$$(3.2) \quad \langle R_0, Sb(A_n) \rangle \notin SCpl_F \quad \text{for every } n > 2.$$

Now we shall prove the Tokarz's theorem on structural completeness of the Łukasiewicz's sentential calculi.

$$(3.3) \quad \langle R_{0^*}, A_n \rangle \in SCpl_F \quad \text{for every } n \geq 2.$$

PROOF. To prove that $\overrightarrow{M_n}(Sb(X)) = \overrightarrow{M_n \times M_2}(Sb(X))$ for every $X \subseteq S$ it suffices to show that $\overrightarrow{M_n}(Sb(X)) = S$ for every $Sb(X) \notin Sat(M_2)$. Suppose that it is not true; i.e. there exists $Sb(X) \notin Sat(M_2)$ such that $\overrightarrow{M_n}(Sb(X)) \neq S$. Hence there exists $v : At \rightarrow |M_n|$ such that $h^v(Sb(X)) \subseteq$

$\{1\}$. Let $e : At \rightarrow S$ be a substitution defined as follows: $e(\gamma) = C\gamma\gamma$. We have $w = h^v e : At \rightarrow \{0, 1\}$ so that $h^w(Sb(X)) = h^v(h^e(Sb(X))) \subseteq h^v(Sb(X)) \subseteq \{1\}$. Contradiction. Hence: $\overrightarrow{M_n \times M_2}(Sb(X)) = \overrightarrow{M_n}(Sb) = Cn(R_0, Sb(A_n) \cup Sb(X)) = Cn(R_{0^+}, A_n \cup X)$. From (1.6) and (3.1) follows that $\langle R_{0^+}, A_n \rangle \in SCpl$.

We shall complete the paper by the following theorem:

(3.4) $\langle R_0, Sb(A_n^p) \rangle \in SCpl$ for every $n \geq 2$.

PROOF. The proof is similar to that (3.1). Using the notation from (3.1) we have $\alpha_x^1 \in S^p$ for every $x \in |M_n|$ and a mapping $g(x) = [\alpha_x^1, g : |M_n| \rightarrow S^p / \approx_n$ is a monomorphism of $\overrightarrow{M_n^p}$ and $\overrightarrow{L_n^p}$. Hence, $\overrightarrow{M_n^p} \geq \overrightarrow{L_n^p} / \approx_n = \overrightarrow{L_n^p}$. From (1.4) and (1.5) we obtain $\overrightarrow{M_n^p} \leq \overrightarrow{L_n^p}$. Thus $\overrightarrow{M_n^p} = \overrightarrow{L_n^p} \in SCpl$.

This theorem is a generalization of some theorem from [6], where it is proved that pure implicative Łukasiewicz's calculi are $SCpl_F$.

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