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STRONGLY FINITE LOGICS: FINITE AXIOMATIZABILITY AND THE PROBLEM OF SUPREMIUM

This is an extended version of a lecture read at the meeting organized by the Łódź section of the Philosophical Society on January 20, 1979. Extended fragments of this paper will appear in "Reports on Mathematical Logic".

This paper, which in its subject matter goes back to works on strongly finite logics (e.g. [8], [9]), is concerned with the following problems:

(1) Let Cn_1, Cn_2 be two strongly finite logics over the same propositional language. Is the supremum of Cn_1 and Cn_2 (noted as $Cn_1 \cup Cn_2$) also a strongly finite operation?

(2) Is any finite matrix (or more precisely, the content of any finite matrix) axiomatizable by a finite set of standard rules?

The first question can be found in [9] (and also in [11]). The second conjecture was formulated by Wolfgang Rautenberg, but investigations into this problem had been carried out earlier in works of many logicians (e.g. the known theorem of Mordchaj Wajsberg [7], see also [5]). Moreover, Stephen Bloom [1] posed a conjecture stronger than (2) that: the consequence determined by a finite matrix (a strongly finite consequence, see [9]) is finitely based, i.e. it is the consequence generated by a finite set of standard rules. This hypothesis was, however, disproved by Andrzej Wroński [10] (and also by Alasdair Urquhart [6]).

In the present paper it is shown that neither (1) nor (2) holds true. The negative answer to (2) can be viewed as a generalization of the result given by Andrzej Wroński [10] (or by [6]).

Let $\underline{S}_0 = (S_0, \circ)$ be the algebra of the propositional language determined by the set $V = \{p_i; i = 0, 1, 2, \ldots\}$ of propositional variables and by a two-argument connective $\circ$. By h^e we denote the extension of the mapping $e : V \to S_0$ to an endomorphism of $\underline{S}_0$ ($h^e \in Hom(\underline{S}_0, \underline{S}_0)$). The symbol $V(\alpha)$ stands for the set of all variables occurring in the formula $\alpha \in S_0$. Moreover, $V(X) = \bigcup\{V(\alpha) : \alpha \in X\}$ for every set $X \subseteq S_0$. The length of a formula is defined as follows:

Definition 1.1.

(i) $l(p_i) = 1$ for every $p_i \in V$
(ii) $l(\alpha \circ \beta) = 1 + l(\alpha) + l(\beta)$ for every $\alpha, \beta \in S_0$
(iii) $l(X) = max\{l(\alpha) : \alpha \in X\}$ for every finite set $X \subseteq S_0$

Let us take into consideration the following three matrices: $K = (\{0,1\}, \{1\}, f^+)$ (the matrix of the classical disjunction), $L = (\{0, a, 1\}, \{a, 1\}, f^=)$ and $M = (\{0, a, 1\}, \{a, 1\}, f^*)$, where

f^+	0	1
0	0	1
1	1	1

$f^=$	0	a	1
0	1	0	0
a	0	0	1
1	0	1	1

f^*	0	a	1
0	0	0	0
a	0	0	0
1	0	0	a

The structural consequences determined by these matrices (or the so-called matrix consequences, see [3]) will be designated by C_K, C_L, C_M. We can easily make the following observation:

(a) $\alpha \in (C_K \cap C_M)(\alpha \circ \alpha)$ for every $\alpha \in S_0$, where $(C_K \cap C_M)(X) = C_K(X) \cap C_M(X)$ for $X \subseteq S_0$
(b) $\beta \in (C_L \cap C_M)(\alpha, \alpha \circ \beta) = C_L(\alpha, \alpha \circ \beta) \cap C_M(\alpha, \alpha \circ \beta)$ for every $\alpha, \beta \in S_0$.
(c) $C_M(\alpha) = S_0$ if $\alpha \in S_0$ and $l(\alpha) > 3$.

Let us take $X_0 = \{p_i \circ p_j; i \neq j\}$ and note that:

(d) $C_M(X_0) \neq S_0$
(e) $V \cap C_K(X_0) = 0$ – it suffices to consider, for every $p_i \in V$, a homomorphism $h_i \in Hom(\underline{S}_0, alg(M))$ such that $h_i(p_j) = 1$ iff $i \neq j$.
(f) $p_i \circ p_i \notin C_K(X_0)$ for every $p_i \in V$ – by (a) and (e).

(g) $V \cap C_L(X_0) = 0$ – let us consider a homomorphism $h \in Hom(\underline{S}_0, alg(L))$ such that $h(p_i) = 0$ for every $p_i \in V$. Then $h(X) \subseteq \{1\}$ and $h(V) \subseteq \{0\}$.

(h) $p_i \circ p_i \notin C_L(X_0)$ for every $p_i \in V$ – if $h_i \in Hom(\underline{S}_0, alg(L))$ is a homomorphism such that $h_i(p_j) = 1$ for $j \neq i$ and $h_i(p_i) = a$, then $h_i(p_i \circ p_i) = f^{=}(a, a) = 0$ and $h_i(p_i \circ p_j) = f^{=}(a, 1) = 1$ for every $j \neq i$.

It immediately follows from (c) and (d) that: if $\alpha \in C_K \cap C_M(X_0)$, then $3 < l(\alpha)$. Hence, by (e) and (f), we get

(i) $(C_K \cap C_M)(X_0) = C_K(X_0) \cap C_M(X_0) = X_0$

Similarly, by (c), (d), (g) and (h), we obtain

(j) $(C_L \cap C_M)(X_0) = C_L(X_0) \cap C_M(X_0) = X_0$

We state without proofs the following easy lemmas:

LEMMA 1.2. *Let Let Cn_1 and Cn_2 be consequence operations on S_0 and let $Cn_1 \cup Cn_2$ be the supremum (the least upper bound in the lattice of all consequences over S_0, see [8]) of Cn_1 and Cn_2. Then, for every set $X \subseteq S_0$,*

$$(Cn_1 \cup Cn_2)(X) = \bigcap\{Y; Y = Cn_1(Y) = Cn_2(Y) = Y \supset X\}.$$

LEMMA 1.3. *Let $\mathbb{K}$ be a class of finite matrices for $\underline{S}_0$ and let $C_{\mathbb{K}}$ be the structural consequence operation determined by $\mathbb{K}$ (that is $C_{\mathbb{K}}(X) = \bigcap\{C_N(X); N \in \mathbb{K}$, see [8]). Then*

$$\alpha \in C_{\mathbb{K}}(X) \equiv \forall_k \forall_{e:V \to \{p_1,\ldots,p_k\}} h^e(\alpha) \in C_{\mathbb{K}}(h^e(X))$$

for every $\alpha \in S_0$ and $X \subseteq S_0$.

The above lemma is closely similar to the criterion of strong finiteness given by Ryszard Wójcicki [8].

THEOREM 1.4. *The consequence $Cn = (C_K \cap C_M) \cup (C_L \cap C_M)$ is not strongly finite and, what is more, $Cn \neq C_{\mathbb{K}}$ for any class $\mathbb{K}$ of finite matrices.*

PROOF. Suppose that $e : V \to \{p_1, \ldots, p_k\}$. Since the set $\{p_1, \ldots, p_k\}$ is finite, there exist $p_i, p_j \in V$ such that $i \neq j$ and $e(p_i) = e(p_j)$. Thus

$h^e(p_i \circ p_i = e(p_i) \circ e(p_i) = e(p_i) \circ e(p_j) = h^e(p_i \circ p_j) \in h^e(X_0) \subseteq Cn(h^e(X_0))$ and hence, by (a), $e(p_i) \in Cn(h^e(X_0))$ for some $p_i \in V$. On the other hand $e(p_i) \circ e(p_k) = h^e(p_i \circ p_k) \in h^e(X_0) \subseteq Cn(h^e(X_0))$ for every $p_k \neq p_i$. So it follows from (b) that $e(p_k) \in Cn(h^e(X_0))$ for every $p_k \in V$. Consequently, $h^e(V) \subseteq Cn(h^e(X^0))$ for every $e : V \to \{p_1, \ldots, p_k\}$.

Let us assume, to the contrary, that $Cn = C_{\mathbb{K}}$ for some class $\mathbb{K}$ of finite matrices. Then, by Lemma 1.3, $V \subseteq Cn(X_0)$. But on the other hand, by (i), (j) and Lemma 1.2, $Cn(X_0) = X_0$. Hence $V \subseteq X_0$, which is a contradiction. $\square$

According to the definition of a strongly finite consequence (see [8]) the operations $C_K \cap C_M$, $C_L \cap C_M$ are strongly finite. Therefore conjecture (1) has been disproved.

Instead of $C_K \cap C_M$, $C_L \cap C_M$ one can take $C_{K \times M}$, $C_{L \times M}$ and by the similar argument it can be shown that $C_{K \times M} \cup C_{L \times M}$ is not a strongly finite consequence. Note that $K \times M$ and $L \times M$ are elementary matrices. It can also be proved that $C_{K \times M} \cup C_{L \times M}$ is uniform, that is, there exists an elementary matrix which is strongly adequate for $C_{K \times M} \cup C_{L \times M}$ (this is the answer to the question posed by Wiesław Dziobiak).

Obviously the set $Cn(0)$ is empty, but when we extend the language $\underline{S}_0$ (and also the matrices K, L, M) by adding some new connectives, then we can easily obtain two strongly finite consequences Cn_1 and Cn_2 such that $Cn_1 \cup Cn_2$ is not strongly finite and $Cn_1 \cup Cn_2(0)$ is not empty.

Theorem 1.4 states that the set of strongly finite logics does not form a sublattice of the lattice of all logics on S_0. From this statement, by an easy verification, the following theorem may also be deduced.

THEOREM 1.5. *The set of all strongly finite logics does not form a lattice.*

It was proved in Theorem 1.4 that the supremum $(C_K \cap C_M) \cup (C_L \cap C_M)$ of two strongly finite logics does not have the strongly finite model property (the notion introduced by Ryszard Wójcicki). In particular, this means that strengthenings of a given strongly finite consequence need not be characterized by finite matrices (need not have the strongly finite model property). This result was first obtained by Wiesław Dziobiak [2].

§.2

Let us proceed to the second conjecture. Further we will consider formulae of the form:

$(*)$ $\gamma_1 \circ (\gamma_2 \circ \ldots \circ (\gamma_{n-1} \circ \gamma_n))$ where $\gamma_1, \gamma_2, \ldots, \gamma_n \in V$.

The following definition is accepted:

DEFINITION 2.1. The set $F(\beta)$, for every $\beta \in S_0$, is defined as follows:

(i) $\beta \in F(\beta)$
(ii) if $\alpha \in F(\beta)$ and if $\gamma \in V$, then $\gamma \circ \alpha \in F(\beta)$
(iii) A formula α belongs to $F(\beta)$ if it can be shown to be in $F(\beta)$ on the basis (i) and (ii).

Moreover, let us define an operation $p : S_0 \to V$:

(i) $p(\gamma) = \gamma$ for every $\gamma = V$
(ii) $p(\alpha \circ \beta) = p(\beta)$ for every $\alpha, \beta \in S_0$

For every $\alpha \in S_0$ and for every $\gamma \in V$, the number of occurrences of the variable γ in the formula α will be denoted by $ind(\alpha, \gamma)$, that is $ind(p_i, p_j) = 0$ if $i \neq j$, $ind(p_i, p_i) = 1$ and $ind(\alpha \circ \beta, \gamma) = ind(\alpha, \gamma) + ind(\beta, \gamma)$. It is easy to see that $F = \bigcup\{F(\gamma); \gamma \in V\}$ is the set of all formulae of the form $(*)$ and that $p(\alpha)$, for $\alpha \in F$, is the initial variable of α (i.e. $p(\alpha) = \gamma_n$ in $(*)$). We quote without proofs:

LEMMA 2.2. *For every formulae $\alpha, beta \in S_0$ and every mapping $e : V \to S_0$*

(i) $l(\alpha) = 2(\sum\limits_{\gamma \in V} ind(\alpha, \gamma)) - 1$
(ii) $\alpha \in F(\beta) \Rightarrow l(\beta) \leqslant l(\alpha) \wedge V(\beta) \subseteq V(\alpha)$,
(iii) $F(\alpha) \cap F(\beta) \neq 0 \Rightarrow F(\alpha) \subseteq F(\beta) \vee F(\beta) \subseteq F(\alpha)$,
(iv) If $h^e(\alpha) \in F$ *and if* $l(h^e(\alpha)) > l(\alpha)$, *then*
  (a) $ind(\alpha, p(\alpha)) = 1$
  (b) $e(p(\alpha)) \in F \wedge \alpha \in F$
  (c) $e(V(\alpha)\backslash\{p(\alpha)\}) \subseteq V$.
(v) If $\alpha \in F \wedge e(P(\alpha)) \in F(\beta)$ *and if the above conditions (a), (c) are fulfilled, then* $h^e(\alpha) \in F(\beta)$.

We will consider the matrix $N = (\{0, 1, 2, 3, 4\}, \{1, 2, 3, 4\}, f)$ where

f	0	1	2	3	4
0	4	4	4	4	4
1	0	2	2	0	4
2	4	4	4	4	4
3	4	0	0	4	4
4	4	4	4	4	4

The designated values in N are $\{1, 2, 3, 4\}$ and hence the set of formulae valid in N can be defined as:

$$E(N) = C_N(0) = \{\alpha \in S_0; h(\alpha) \neq 0 \text{ for every } h \in Hom(\underline{S}_0, alg(N))\}$$

It will be proved that $E(N)$ is not finitely axiomatizable by means of standard rules. Let us recall that a rule $r \subseteq 2^{S_0} \times S_0$ is said to be standard (polynomial c.f. [3]) if there exist a finite set $X \subseteq S_0$ and a formula $\alpha \in S_0$ such that $r = r_\alpha^I$, where

$$r_\alpha^X = \{(h^e(X), h^e(\alpha)); e : V \to S_0\}.$$

Observe that rules of the form r_α^0, where 0 is the empty set, are also standard. Such rules are called axiomatic. Given a set R of rules we shall write $Cn(R, X)$ (or $C_R(X)$) to denote the least superset of X closed under R. We say that the matrix N is axiomatizable by a set R of rules if and only if $R(N) = Cn(R, 0)$.

LEMMA 2.3. *For every formula* $\alpha \in S_0 : \alpha \notin E(N) \equiv \alpha \in F \wedge ind(\alpha, \gamma) = 1$ *for some* $\gamma \in V$.

THEOREM 2.4. *The matrix N cannot be axiomatized by a finite set of standard rules.*

PROOF. Suppose to the contrary that $E(N) = Cn(R, 0)$ for some finite set R of standard rules. Thus the members of R are unfailing in the matrix N that is:

(a) $r \in R \wedge (X, \Phi) \in r \wedge X \subseteq E(N) \Rightarrow \Phi \in E(N)$.

Since the set R is finite, there exists a natural number k such that

(b) $k = max\{l(X, \alpha); r^X \in R\}$.

Let us take $\alpha_0 = p_0$ and $\alpha_{n+1} = p_{n+1} \circ \alpha_n$ for every natural number n. It is obvious that:

(c) $V(\alpha_n) = \{p_0, p_1, \ldots, p_n\}$,
$l(\alpha_n) = 2n + 1$,
$\alpha_n \in F(p_0) \subseteq F$.

Moreover, we shall prove that:

(d) $\Phi \in F(\alpha_n) \cap E(N) \equiv l(\Phi) \geqslant 4n + 3$,
$F(\alpha_n) \cap E(N) \neq 0$.

If $\Phi \in F(\alpha_n) \cap E(N)$, then it follows from Lemma 2.2 (ii) that $\{p_0, \ldots, p_n\} \subseteq V(\Phi)$ and therefore, according to Lemma 2.3, $ind(\Phi, p_i) \geqslant 2$ for $i = 0, \ldots, n$. Hence, by 2.2 (i), $l(\Phi) \geqslant 4n + 3$. To show $E(N) \cap F(\alpha_n) \neq 0$ it suffices to consider the formula $\Psi = \alpha_n(p_0/\alpha_n)$. From Lemma 2.2 (v) it follows that this formula is an element of $F(\alpha_n)$ and according to 2.3, 2.1: $p_0 \circ \Psi \in F(\alpha_n) \cap E(N)$.

Let us assume that the sequence:

(*e*) $\Phi_1, \Phi_2, \ldots, \Phi_m$

is a proof of a formula $\Phi \in E(N) \cap F(\alpha_k)$ (where the natural number k is defined in (a)) on the ground of the rules R, i.e. $\Phi_m = \Phi$ and for every $i \leqslant m$ there exist a rule $r \in R$ and a set $Y \subseteq \{\Phi_1, \ldots, \Phi_{i-1}\}$ such that $(X, \Phi_i) \in r$. Suppose $m = 1$. Then $\Phi = h^e(\alpha)$ and $r^0 \in R$ for some $\alpha \in S_0$. But $l(\alpha) < k < 4k + 3 < l(\Phi)$ by (b) and (d). Hence, it follows from Lemma 2.2 (iv) that $\alpha \in F$ and $ind(\alpha, p(\alpha)) = 1$. Consequently $\alpha \notin E(N)$ by Lemma 2.3, which contradicts assumption (a). Assume that no formula in $F(\alpha_k) \cap E(N)$ has a proof on the ground of R with less than m elements (where $m \geqslant 2$). Since $\Phi_m = \Phi$, there exists $X \subseteq S_0$, $\alpha \in S_0$ and mapping $e : V \to S_0$ such that $h^e(X) \subseteq \{\Phi_1, \ldots, \Phi_{m-1}\} \subseteq E(N)$, $h^e(\alpha) = \Phi$, and $r^X_\alpha \in R$. Moreover, by (b) and (d), $l(\alpha) < l(\Phi)$. Thus, according to Lemma 2.2 (iv):

(f) $\alpha \in F$,
$ind(\alpha, p(\alpha)) = 1$,
$e(V(\alpha) \backslash \{p(\alpha)\}) \subseteq V$

On the other hand $e(p(\alpha)) \subset F(p(\alpha))$ by 2.1 and hence $h^e(\alpha) = \Phi \in F(e(p(\alpha)))$ – see Lemma 2.2 (v). Since $\Phi \in F(\alpha_k)$, it follows from 2.2 (ii), (iii) that

(g) $F(e(p(\alpha))) \subseteq F(\alpha_k)$

Let us consider a substitution $f : V \to S_0$ such that:

$$f(\gamma) = \begin{cases} (p_0 \circ p_0) \circ p_0 & \text{if} \quad \gamma \notin V(\alpha) \\ p_1 \circ p_0 & \text{if} \quad \gamma = p(\alpha) \\ p_0 & \text{if} \quad \gamma \in V(\alpha) \backslash \{p(\alpha)\}. \end{cases}$$

It follows from 2.2 (v) and 2.3 that $h^f(\alpha) \notin E(N)$. Since $r_\alpha^X \in R$, there exists $\Psi \in X$ such that $h^f(\Psi) \notin E(N)$. It can be proved using 2.3, (f), (e) that

(h) $V(\Psi) \subseteq V(\alpha)$,
$\Psi \in F$ and $p(\Psi) = p(\alpha)$,
$ind(\Psi, p(\Psi)) = 1$.

The simple conclusion based on (h), (f) and Lemma 2.2 (iv) is that $h^e(\Psi) \in F(e(p(\alpha)))$. Hence, by (g), $h^e(\Psi) \in F(\alpha_k)$ – which contradicts the inductive hypothesis because $h^e(\Psi)$ has a proof with less than m elements. It has been proved, by induction on the number of elements in a proof of a formula $\Phi \in F(\alpha_k) \cap E(N)$, that $Cn(R, 0) \cap F(\alpha_k) \cap E(N) = 0$. Consequently, by (d), $Cn(R, 0) \neq E(N)$, which completes the proof of our theorem. □

§3.

Finally we shall deal with the following problem: under what condition on strongly finite logics do the questions considered have positive answers? In the sequel some solution of this problem is presented.

Let $\underline{S} = (S, \equiv, +, F_1, \ldots, F_n)$ be a propositional language based on the set $V = \{p_0, p_1, \ldots, \}$ of propositional variables. We will assume that $+$ and $\equiv$ are two-argument connectives. A consequence operation Cn on S is said to be a disjunctive one if and only if $Cn(X, \alpha) \cap Cn(X, \beta) = Cn(X, \alpha + \beta)$ for every $X \subseteq S$ and every $\alpha, \beta \in S$ ($Cn \in D$, see [5]). We say that Cn is a consequence with the identity connective ($Cn \in I$, see [4]) when the binary relation on S defined as follows:

$$\alpha \approx \beta \text{ iff } \alpha \equiv \beta \in Cn(0)$$

is a congruence in $\underline{S}$ consistent with Cn, that is

$$\alpha \approx \beta \ \Rightarrow \ Cn(\alpha) = Cn(\beta)$$

If Cn is a disjunctive operation with identity, then we will write $Cn \in DI$.

LEMMA 3.1 $Cn_1, Cn_2 \in DI \Rightarrow Cn_1 \cup Cn_2 \in DI$

LEMMA 3.2. *If $Cn \in DI$ is a strongly finite consequence, then $\{Cn_1 \in Struct; Cn \leqslant Cn_1 \in D\}$ is a finite subset of the set of strongly finite consequences.*

THEOREM 3.3. *If $Cn_1, Cn_2 \in DI$ are strongly finite consequences, then $Cn_1 \cup Cn_2$ is also a strongly finite operation.*

PROOF. Since $Cn_1, Cn_2 \in DI$, it follows from 3.1 that $Cn_1 \cup Cn_2 \in DI$ and, which is obvious, $Cn_1 \leqslant Cn_1 \cup Cn_2$. Thus $Cn_1 \cup Cn_2$ is a strongly finite consequence by Lemma 3.2. □

It can also be proved that:

THEOREM 3.4. *If $Cn \in DI$ is a strongly finite consequence, then Cn is finitely based, that is, $Cn = C_R$ for a finite R of standard rules.*

It conclusion it is worth while adding, as was mentioned after the lecture, that Jan Zygmunt had proved that: every disjunctive consequence determined by an elementary finite matrix is finitely based.

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