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Krzysztof Szymanek

ON INFORMATION FUNCTIONS
PART ONE: BASIC FORMAL PROPERTIES

In this paper we present the definition and fundamental properties of the information functions – functions which establish a correspondence between sets of formulas and information contained in these sets. The intuitions of the notion of information stem from Bar-Hillel–Carnap's conception derived from [1]. The information contained in the proposition is, according to this conception, the set of all state-descriptions excluded by this proposition.

By S we shall note the set of all formulas in the language $\{\neg, \wedge, \vee, \rightarrow\}$ and by C, the classical consequence operation over S. Instead of $C(\emptyset)$ we write $Taut$. The set of all complete subsets of S we note as Cpl.

We put forward the following definition:

Definition A.1. A function $J : P(S) \rightarrow Z$, where Z is a family of sets, is called an information function iff

(J1) $J(C(X)) \subseteq J(X)$, for any $X \subset S$
(J2) $J(X) = \bigcup\{J(\{\alpha\}) : \alpha \in X\}$, for any $X \subset S$
(J3) $J(\{\alpha\}) \cap J(\{\neg\alpha\}) = \emptyset$, for any $\alpha \in S$

The condition (J2) of the above definition may seem to be non-intuitive; it may seem, for example that the collective information contained in the formulas p and $p \rightarrow q$ is less than the information contained in the set $\{p, p \rightarrow q\}$. However, according to the interpretation presented above, it is not the case:

Let A be a set; we call it a set of all possible cases. For every $\underline{a} \in A$ we assume that $\underline{a}$ is the case of the following kind: "in the time t, in the place r, x occurred (is occurring or will occur)".

The state-description is every function $f \in \widehat{W}$, where $\widehat{W} = \{0,1\}^A$. If $\underline{a} \in A$ and $f \in \widehat{W}$, then notation $f(\underline{a}) = 1$ means that the case $\underline{a}$ occurred (is occurring or will occur); notation $f(\underline{a}) = 0$ means that the case $\underline{a}$ did not occur (is not occurring, will not occur). To every proposition p we attribute its content, i.e. an element $\underline{a}_p$ of the set A. The information contained in p is the set $I_{\{p\}} = \{f \in \widehat{W} : f(\underline{a}_p) = 0\}$. Similarly: $I_{\{p \to q\}} = \{f \in \widehat{W} : f(\underline{a}_p) = 1 \text{ and } f(\underline{a}_q) = 0\}$. The sets $\{p, p \to q\}$ and $\{p, q\}$, as logically equivalent, contain the same information, namely:

$$I_{\{p,p\to q\}} = I_{\{p,q\}} = \{f \in \widehat{W} : f(\underline{a}_p = 0 \text{ or } f(\underline{a}_q) = 0\}.$$

It is easy to see, that $I_{\{p,p\to q\}} = I_{\{p\}} \cup I_{\{p\to q\}}$. □

The following lemma gives several primary properties of the information functions. Instead of $J(\{\alpha_1, \ldots, \alpha_n\})$ we shall write $J(\alpha_1, \ldots, \alpha_n)$.

LEMMA A.2. *If J is an information function, then:*

(a) $J(\emptyset) = \emptyset$
(b) *if* $X \subseteq Y$, *then* $J(X) \subseteq J(Y)$, *for any* $X, Y \subseteq S$
(c) $J(C(X)) = J(X)$, *for any* $X \subseteq S$
(d) *if* $\mathcal{R} \subseteq P(S)$, *then* $J(\bigcup \mathcal{R}) = \bigcup\{J(X) : X \in \mathcal{R}\}$
(e) *if* $\emptyset \neq \mathcal{R} \subseteq P(S)$, *then* $J(\bigcap \mathcal{R}) \subseteq \bigcap\{J(X) : X \in \mathcal{R}\}$
(f) *if* $X, Y \subseteq S$ *are theories, then* $J(X) \cap J(Y) = J(X \cap Y)$
(g) $J(Taut) = \emptyset$
(h) *if* $\alpha \to \beta \in Taut$, *then* $J(\beta) \subseteq J(\alpha)$, *for any* $\alpha, \beta \in S$
(i) $J(\neg\alpha) = J(S) \backslash J(\alpha)$, *for any* $\alpha \in S$
(j) $J(\alpha \wedge \beta) = J(\alpha) \cup J(\beta)$, *for any* $\alpha, \beta \in S$
(k) $J(\alpha \vee \beta) = J(\alpha) \cap J(\beta)$, *for any* $\alpha, \beta \in S$
(l) $J(\alpha \to \beta) = J(\neg\alpha) \cap J(\beta)$, *for any* $\alpha, \beta \in S$

Let W be a set. By $Inf(W)$ we denote the class of all information functions J such that $J(S) = W$. From the item (b) of the above lemma it follows that if $J \in Inf(W)$, then $J : P(S) \to P(W)$.

THEOREM A.3. *Assume that* $J : P(S) \to Z$ *and* $J(X) = \bigcup\{J(\alpha) : \alpha \in X\}$, *for any* $X \subseteq S$. *Let* $W = J(S)$. *Then* $J \in Inf(W)$ *iff the mapping* $h : S \to P(W)$ *defined by*

$$h(\alpha) = W \backslash J(\alpha), \text{ for every } \alpha \in S$$

is a homomorphism, i.e. $h(\neg\alpha) = W\backslash h(\alpha), h(\alpha \wedge \beta) = h(\alpha) \cap h(\beta), h(\alpha \vee \beta) = h(\alpha) \cup h(\beta), h(\alpha \rightarrow \beta) = h(\neg\alpha) \cup h(\beta)$, *for any* $\alpha, \beta \in S$. □

Let $J \in Inf(W), B_J = \{J(\alpha) : \alpha \in S\}, \underline{B}_J = < B_J, W, 0, -, \cap, \cup, \rightarrow>$, where $\cap$ and $\cup$ are the usual set-theoretical operations and $-U = W\backslash U, U \rightarrow V = -U \cup V$, for any $U, V \in B_J$. Besides, we define:

$$C_J(X) = \{\alpha \in S : J(\alpha) \subseteq J(X)\}, \text{ for every } X \subseteq S$$

C_J is a consequence operation and for any $X \subseteq S : C(C_J(X)) = C_J(X)$.

THEOREM A.4. *If* $J \in Inf(W)$, *then* $\underline{B}_J$ *is isomorphic with the Lindenbaum algebra* $S/C_J(\emptyset)$. □

Let $J \in Inf(W)$. For every $U \subseteq W$ we define:

$$con_J(U) = \{\alpha \in S : J(\alpha) \subseteq U\}$$

LEMMA A.5. *If* $J \in Inf(W)$, *then:*
(a) $C(con_J(U)) = C_J(con_J(U)) = con_J(U)$, *for any* $U \subseteq W$
(b) $con_J(J(X)) = C_J(X)$, *for any* $X \subseteq S$
(c) $C_J(X) = C_J(Y)$ *iff* $J(X) = J(Y)$, *for any* $X, Y \subseteq S$
(d) if $\emptyset \neq \mathcal{R} \subseteq P(W)$, *then* $con_J(\bigcap \mathcal{R}) = \bigcap\{con_J(U) : U \in \mathcal{R}\}$
(e) $con_J(X\backslash\{x\}) \in Cpl$, *for any* $x \in W$ □

If $J \in Inf(W)$, then the family $T_J = \{J(X) : X \subseteq S\}$ is a topology in the set W. We say that topological space $< W, T_J >$ is determined by J. $< W, T_J >$ is a zero-dimensional space with countable basis B_J. In addition: $intU = J(con_J(U))$, for any $U \subseteq W$.

If for any $\alpha, \beta \in S : J(\beta) \subseteq J(\alpha)$ implies $\alpha \rightarrow \beta \in Taut$, then we write $J \in Inf^+(W)$. If $C = C_J$ then we write $J \in Inf^0(W)$. Observe that $J \in Inf^+(W)$ iff $con_J(\emptyset) = Taut$. It is simple that $Inf^0(W) \subseteq Inf^+(W)$, for any W.

LEMMA A.6. *If* $J \in Inf^+(W)$, *then*

$J \in Inf^0(W)$ *iff the space* $<W, T_J>$ *is compact.* □

Note, that there exist a set W and $J \in Inf^+(W)$, such that $J \notin Inf^0(W)$.

LEMMA A.7. *If $J \in Inf(W)$, then there are $W_H \subseteq W$ and $J_H \in Inf(W_H)$ such that:*

(a) $J_H(X) = J(X) \cap W_H$, *for any* $X \subseteq S$
(b) $C_{J_H} = C_J$
(c) W_H *is a dense subset of* W
(d) $<W_H, T_{J_H}>$ *is a subspace of* $<W, T_J>$
(e) $<W_H, T_{J_H}>$ *is a Hausdorff space.* □

If $J \in Inf(W)$ and $< W, T_J >$ is a Hausdorff space, then we write $J \in Inf_H(W)$.

LEMMA A.8. *Let $J \in Inf(W)$. The mapping $f : W \to P(S)$ we define by:*

$$f(x) = con_J(W \backslash \{x\}), \text{ for every } x \in W$$

Then we have:

(a) $f[W] \subseteq Cpl$
(b) $f[W] = Cpl$ *iff* $J \in Inf^0(W)$
(c) f *is one-to-one iff* $J \in Inf_H(W)$. □

LEMMA A.9.

(a) *if* $Inf_H(W) \neq \emptyset$, *then* $\overline{\overline{W}} \leq \mathsf{c}$
(b) *if* $Inf_H^+(W) \neq \emptyset$, *then* $\overline{\overline{W}} \geq \omega$
(c) $Inf_H^0(W) \neq \emptyset$ *iff* $\overline{\overline{W}} = \mathsf{c}$.

Sketch of proof of (c): Necessity results from Lemma A.8., in view of $\overline{\overline{Cpl}} = \mathsf{c}$. The function $J : P(S) \to P(Cpl)$ defined by:

$$J(X) = \{Z \in Cpl : X \not\subseteq Z\}, \text{ for every } X \subseteq S$$

is an element of $Inf_H^0(Cpl)$. □

Let $J_1 \in Inf_H(W_1)$ and $J_2 \in Inf_H(W_2)$. A mapping $h : W_1 \to W_2$ is called an isomorphism from J_1 into J_2 iff h is one-to-one and onto and $J_2(X) = h[J_1(X)]$, for any $X \subseteq S$.

J_1 and J_2 are said to be isomorphic ($J_1 \approx J_2$) iff there is an isomorphism from J_1 onto J_2. Note that each isomorphism from J_1 onto J_2 is a homeomorphism of the spaces $<W_1, T_{J_1}>$ and $<W_2, T_{J_2}>$. There exists at most one isomorphism from J_1 onto J_2.

THEOREM A.10. *If $J_1 \in Inf_H^0(W_1)$ and $J_2 \in Inf_H^0(W_2)$ then $J_1 \approx J_2$.* □

THEOREM A.11. *If $J \in Inf_H(W)$ and $\overline{J} \in Inf_H^0(\overline{W})$, then there exists a unique mapping $h : W \to \overline{W}$ such that:*

(a) h is a homeomorphism from W onto $h[W]$

(b) $h[J(X)] = \overline{J}(X) \cap h[W]$, for any $X \subseteq S$. □

REMARK A.12. If $J \in Inf_H^0(W)$, then $<W, T_J>$ is a zero-dimensional, compact, dense-in-itself, Hausdorff space. It is well known that each such space is homeomorphic with the Cantor space. So we may assume that $<W, T_J>$ is the Cantor space. □

References

[1] Y. Bar-Hillel and R. Carnap [1952], *An Outline of a Theory of Semantic Information*, **Technical Report** No 247. Cambridge (Mass.), MIT Research Laboratory of Electronics. Reprint in: Y. Bar-Hillel [1964] *Language and Information*, Reading (Mass.), Addison-Wesley.

Section of Logic and Methodology
Silesian University
Katowice, Poland