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ON INFORMATION FUNCTIONS PART TWO: APPLICATIONS TO THE LOGIC OF THEORY CHANGE

The purpose of this work is to show how the notion of information function introduced in [1] can be used in the logic of theory change developed by P. Gärdenfors, C. E. Alchourron and D. Makinson in [2], [3] and [4].

The simplest and best known form of theory change is expansion, where a new position, if consistent with a given theory K, is set-theoretically added to K and this expanded set is then closed under logical consequence. Second form is theory contraction, where a proposition α, which was earlier in a theory K, is rejected. The basic problem is to determine which propositions should be rejected along with α so that the contracted theory will be closed under logical consequence. Third kind of change is revision, where a proposition, in general inconsistent with a given theory K, is added to K under the requirement that the revised theory be consistent and closed under logical consequence.

In this note we shall focus on the contraction functions, i.e. functions which reflect the process of contraction according to Gärdenfors postulates (see definition B.1.). We will point how using the notions presented in Part One can be proved a few important theorems about contraction functions (in the less general case than in [2] and [3], where C is not necessarily the classical consequence).

Our leading idea is rather intuitive. Loosely speaking, a proposition α is rejected from a theory K if at least a part of information contained in α has lost its credibility and the contraction operation is to reject, along with α, those formulas which contain any component of "bad" information.

In this part by J we shall denote the "unique" (see [1], theorem A.10.) element of the set $Inf^0_H(W)$, where $<W, T_J>$ is the Cantor space (see [1], remark A.12.). Instead of $con_J(U)$ we write $con(U)$. The formulas of the form $(\alpha \to \beta) \wedge (\beta \to \alpha)$ will be abbreviated as $\alpha \leftrightarrow \beta$. From now on we assume that K is any fixed theory.

Definition B.1. (cf. [2], [3], [4]) A function $K\dot{-} : S \to P(S)$ is called a contraction function over K iff for every $\alpha, \beta \in S$:

(c1) $K\dot{-}\ \alpha$ is a theory
(c2) $K\dot{-}\ \alpha \subseteq K$
(c3) if $\alpha \notin K$, then $K\dot{-}\ \alpha = K$
(c4) if $\alpha \notin Taut$, then $\alpha \notin K\dot{-}\ \alpha$
(c5) if $\alpha \leftrightarrow \beta \in Taut$, then $K\dot{-}\ \alpha = K\dot{-}\ \beta$
(c6) $K \subseteq C((L\dot{-}\ \alpha) \cup \{\alpha\})$

Lemma B.2. *A function $K\dot{-} : S \to P(K)$ is a contraction function over K iff there exists a mapping $F : S \to P(W)$ such for every $\alpha, \beta \in S$:*

(a) $F(\alpha) \neq \emptyset$ *iff* $\alpha \in K \backslash Taut$
(b) $F(\alpha) \subseteq J(\alpha)$
(c) if $\alpha \leftrightarrow \beta \in Taut$, *then* $F(\alpha) = F(\beta)$
(d) $K\dot{-}\ \alpha = con(J(K) \backslash F(\alpha))$ □

Any mapping satisfying (a)–(d) above is called a determinant of contraction $K\dot{-}$. For every $\alpha \in S$ the set $F(\alpha)$ represents a part of the information contained in the formula α, which is excluded while contracting the set K into the set $K\dot{-}\ \alpha$. If F is a determinant of $K\dot{-}$, then the mapping $\dot{F}$ defined by $\dot{F}(\alpha) = clF(\alpha)$, for any $\alpha \in S$, is also a determinant of $K\dot{-}$, so called closed determinant. If F_1 and F_2 are two determinants of contraction $K\dot{-}$, then $clF_1(\alpha) = clF_2(\alpha)$, for any $\alpha \in S$. Consequently, each contraction function over K has exactly one closed determinant.

We define (cf. [2]) $K \bot \alpha$ to be the set of all maximal subtheories K' of K such that $\alpha \notin K'$. Note that $\alpha \notin K$ iff $K \bot \alpha = \{K\}$, likewise $K \bot \alpha = \emptyset$ iff $\alpha \in Taut$. For every $\alpha, \beta \in S$ we have: $K \bot \alpha = K \bot \beta$ iff $\alpha \leftrightarrow \beta \in Taut$.

Lemma B.3. *If $\alpha \in K \backslash Taut$, then:*

(a) $con(J(K) \backslash \{x\}) \in K \bot \alpha$ *iff* $x \in J(\alpha)$, *for any* $x \in W$
(b) the mapping $f : J(\alpha) \to K \bot \alpha$ *given by*

$$f(x) = con(J(K) \backslash \{x\}), \text{ for any } x \in J(\alpha)$$

is one-to-one and onto. □

COROLLARY B.4. *If* $\alpha \in K \backslash Taut$, *then* $\overline{\overline{K \bot \alpha}} = \mathsf{c}$.

PROOF. $J(\alpha)$ is a non-empty, closed-open subset of W, hence it is homeomorphic with the Cantor space $<W, T_J>$. Then by the above lemma we have: $\overline{\overline{K\bot\alpha}} = \overline{\overline{J(\alpha)}} = \overline{\overline{W}} = \mathsf{c}$. □

LEMMA B.5. (cf. [2]) *For every* $\alpha \in K\backslash Taut : \bigcap K\bot\alpha = K \cap C(\neg\alpha)$.

PROOF. Suppose $\alpha \in K\backslash Taut$ and let $U_x = J(K)\backslash\{x\}$, for any $x \in J(\alpha)$. By lemma B.3 and lemmas A.2 and A.5 of [1] we have:
$\bigcap K\bot\alpha = \bigcap\{con(U_x) : x \in J(\alpha)\} = con[\bigcap\{U_x : x \in J(\alpha)\}] =$
$= con(J(K)\backslash J(\alpha)) = con(J(K) \cap (W\backslash J(\alpha))) = con(J(K) \cap J(\neg\alpha)) =$
$= con(J(K)) \cap con(J(\neg\alpha)) = K \cap C(\neg\alpha)$. □

We say that γ is a selection function for K (cf. [3], [4]), if
(i) $\gamma : \{K\bot\alpha : \alpha \in S\} \to P(P(K))$
(ii) $\gamma(K\bot\alpha)$ is a non-empty subset of $K\bot\alpha$ whenever $K\bot\alpha \neq \emptyset$
(iii) if $K\bot\alpha = \emptyset$ then $\gamma(K\bot\alpha) = \{K\}$
It is obvious that $\gamma(K\bot\alpha) = \{K\}$ iff $\alpha \in Taut$ or $\alpha \in S\backslash K$.

THEOREM B.6. (cf. [3]) *A function* $K\dot{-} : S \to P(K)$ *is contraction function over* K *iff there exists a selection function* γ *for* K *such that:* $K\dot{-}\ \alpha = \bigcap\gamma(K\bot\alpha)$, *for any* $\alpha \in S$.

PROOF. Assume that $K\dot{-}$ is a contraction function over K and let F be a determinant of $K\dot{-}$. For every $\alpha \in K\backslash Taut$ and $x \in F(\alpha)$ we put $U^\alpha_x = J(K)\backslash\{x\}$ and $\gamma(K\bot\alpha) = \{con(U^\alpha_x) : x \in F(\alpha)\}$. If $\alpha \in S\backslash(K\backslash Taut)$, then let $\gamma(K\bot\alpha) = \{K\}$. The function γ is well-founded, for if $K\bot\alpha = K\bot\beta$ then $\alpha \leftrightarrow \beta \in Taut$ and $F(\alpha) = F(\beta)$, hence $\gamma(K\bot\alpha) = \gamma(K\bot\beta)$. By lemma B.3 γ is a selection function for K. Next we have: $\bigcap\gamma(K\bot\alpha) = \bigcap\{con(U^\alpha_x) : x \in F(\alpha)\} = con(\bigcap\{U^\alpha_x : x \in F(\alpha)\}) = con(J(K)\backslash F(\alpha)) = K\dot{-}\ \alpha$, for any $\alpha \in K\backslash Taut$. If $\alpha \in S\backslash(K\backslash Taut)$, then naturally $\bigcap\gamma(K\bot\alpha) = K = K\dot{-}\ \alpha$.

To prove the converse implication let for every $\alpha \in S : F(\alpha) = J(K)\backslash J(\bigcap\gamma(K\bot\alpha))$, where γ is a selection function for K. We will examine that F satisfies the conditions (a)–(d) of lemma B.2.

(a) From the definition of F and lemma A.5. of [1] we have: $F(\alpha) = \emptyset \Leftrightarrow J(K) = J(\bigcap\gamma(K\bot\alpha)) \Leftrightarrow K = \bigcap\gamma(K\bot\alpha) \Leftrightarrow \gamma(K\bot\alpha) = \{K\} \Leftrightarrow \alpha \in$

$S \backslash K$ or $\alpha \in Taut$. Hence $F(\alpha) \neq \emptyset$ iff $\alpha \in K \backslash Taut$.

(b) If $\alpha \in S \backslash (K \backslash Taut)$ then from (a) follows $F(\alpha) = \emptyset \subseteq J(\alpha)$. Suppose $\alpha \in K \backslash Taut$. By lemma B.5.: $J(K \cap C(\neg\alpha)) \subseteq J(\bigcap \gamma(K \bot \alpha))$ hence $F(\alpha) = J(K) \backslash J(\bigcap \gamma(K \bot \alpha)) \subseteq J(K) \backslash J(K \cap C(\neg\alpha)) = J(K) \backslash (J(K) \cap J(\neg\alpha)) = J(K) \backslash (J(K) \cap (W \backslash J(\alpha))) = J(\alpha)$.

(c) is obvious

(d) $con(J(K) \backslash F(\alpha)) = con[J(K) \backslash (J(K) \backslash J(\bigcap \gamma(K \bot \alpha)))] = con(J(\bigcap \gamma(K \bot \alpha))) = \bigcap \gamma(K \bot \alpha) = K \dot{-} \alpha$. □

We say that a contraction function $K \dot{-}$ is determined by selection function γ iff $K \dot{-} \alpha = \bigcap \gamma(K \bot \alpha)$, for any $\alpha \in S$ (cf. [3]).

Now we shall consider two special kinds of contraction functions. First, we say that contraction function $K \dot{-}$ is a maxichoice contraction function iff selection function γ determining $K \dot{-}$ satisfies condition $\overline{\overline{\gamma(K \bot \alpha)}} = 1$, for any $\alpha \in K \backslash Taut$ (cf [3]). Second, $K \dot{-}$ is a full meet contraction function iff selection γ determining $K \dot{-}$ satisfies $\gamma(K \bot \alpha) = K \bot \alpha$, for any $\alpha \in K \backslash Taut$ (cf. [3]).

Remark B.7. From lemmas B.2. and B.3. it follows that $K \dot{-}$ is a maxichoice contradiction function iff $K \dot{-}$ has exactly one determinant F, for which $\overline{\overline{F(\alpha)}} = 1$, for any $\alpha \in K \backslash Taut$. Likewise, $K \dot{-}$ is a full meet contraction if it has determinant F such that $F(\alpha) = J(\alpha)$, for any $\alpha \in K$. □

The full meet contraction function over K will be denoted as $K \sim$. It easily results from lemma B.5. that $K \sim \alpha = K \cap C(\neg\alpha)$, whenever $\alpha \in K \backslash Taut$.

Theorem B.8. (cf. [2]) *Let $K \dot{-}$ be any contraction function over K. Then*

(a) $K \dot{-}$ is a maxichoice contraction function iff $C((K \dot{-} \alpha) \cup \{\neg\alpha\}) \in Cpl$, for any $\alpha \in K \backslash Taut$

(b) $K \dot{-}$ is a full meet contraction function iff $C((K \dot{-} \alpha) \cup \{\neg\alpha\}) = C(\neg\alpha)$, for any $\alpha \in K \backslash Taut$.

Proof is based on the equation: $C((K \dot{-} \alpha) \cup \{\neg\alpha\}) = con(W \backslash F(\alpha))$, for any $\alpha \in K \backslash Taut$. □

THEOREM B.9. *Let $K\dot{-}$ be any contraction function over K. Then for every $\alpha \in S$ one of the following conditions is satisfied:*

(a) *there exists $\beta \in S$ such that $\alpha \rightarrow \beta \in Taut$ and $K\dot{-}\ \alpha = K\sim\beta$*

(b) *there exists a sequence $\{\beta_i\}_{i\in N}$ such that*

(b_1) $\beta_1 = \alpha, \beta_i \rightarrow \beta_j \in Taut$ *whenever* $i \leq j$

(b_2) $K\sim\beta_i \subseteq K\sim\beta_j$ *whenever* $i \leq j$

(b_3) $K\dot{-}\ \alpha = \bigcup\{K\sim\beta_i : i \in N\}$.

The clause (a), which clearly implies (b), has been separately stated merely to make (b) more intuitive. □

References

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