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CLASSICAL SUBTHEORIES AND INTUITIONISM

By S we shall denote the set of all formulas in the language $\{\neg, \&, \vee, \Rightarrow\}$ and by C the classical consequence over S. The set of all theories we denote as $\mathbf{Th}$. $\mathbf{Th}_0$ is the set of all theories T such that $T = C(\alpha)$, for some $\alpha \in S$. By $\mathbf{Th}_1$, we denote the set $\mathbf{Th} \backslash \mathbf{Th}_0$. The set of all complete theories we denote as $\mathbf{Cpl}$. For a given $T \in \mathbf{Th}$ let $L_T = \{Y \subseteq T : Y \in \mathbf{Th}\}$. It is obvious that for every $T \in \mathbf{Th}$ the system $< L_T, \subseteq >$ is a lattice. It is evident that in this lattice $X \sqcup Y = C(X \cup Y)$ and $X \sqcap Y = X \cap Y$, for any $X, Y \in L_T$.

In [1] and [2] Dzik has proved among others that the lattice $< L_S, \subseteq >$ is an implicative lattice and its content (i.e. the set of all formulas in the language $\{\vee, \&, \Rightarrow, \neg\}$ which are valid in the lattice $< L_S, \sqcup, \sqcap, \Rightarrow, \rightarrow >$) is equal to the set INT of all formulas provable in the intuitionistic calculus. Here we will prove the following two theorems:

THEOREM 1. *If $T \in \mathbf{Th} \backslash \{C(\emptyset)\}$, then $< L_T, \subseteq >$ is an implicative lattice and its content it equal to INT.*

THEOREM 2. *Let $T_0 \in \mathbf{Th}_0 \backslash \{C(\emptyset)\}$ and $T_1 \in \mathbf{Th}_1$. Then for every $T \in \mathbf{Th}$ the lattice $< L_T, \subseteq >$ is isomorphic with one of the following lattices:*

$$< L_{C(\emptyset)}, \subseteq >, \ < L_{T_0}, \subseteq >, \ < L_{T_1}, \subseteq > .$$

We define the function $J : \mathcal{P}(S) \longrightarrow \mathcal{P}(\mathbf{Cpl})$ by:

$$J(X) = \{Z \in \mathbf{Cpl} : X \nsubseteq Z\}, \text{ for any } X \subseteq S.$$

LEMMA 1. (cf. [1], proof of Theorem 27).

(a) $J(C(X)) = J(X)$*, for any* $X \subseteq S$
(b) $J(X) = \bigcup\{J(\alpha) : \alpha \in X\}$*, for any* $X \subseteq S$
(c) $J(\neg\alpha) = \mathbf{Cpl}\backslash J(\alpha)$*, for any* $\alpha \in S$
(d) $J(\alpha\&\beta) = J(\alpha) \cup J(\beta)$*, for any* $\alpha, \beta \in S$
(e) $J(X) = \emptyset$ *iff* $X \subseteq C(\emptyset)$*, for any* $X \subseteq S$
(f) if $\emptyset \neq \mathcal{R} \subseteq \mathcal{P}(S))$*, then* $J(\bigcup\mathcal{R}) = \bigcup\{J(X) : X \in \mathcal{R}\}$
(g) $J(X) \cap J(Y) = J(C(X) \cap C(Y))$*, for any* $X, Y \subseteq S$
(h) if $T_1, T_2 \in \mathbf{Th}$*, then* $T_1 \subseteq T_2$ *iff* $J(T_1) \subseteq (T_2)$*.* □

Note that from (a), (b) and (c) of the above Lemma it follows that the function J is an information function in the sense of [5]. Observe that by virtue of (e), (c), (f) and (g) of Lemma 1 the family $\mathcal{T} = \{J(X) : X \subseteq S\}$ is a topology in the set **Cpl**. By $\mathcal{W}$ we denote the topological space $< \mathbf{Cpl}, \mathcal{T} >$. If $\mathcal{Z} \subseteq \mathbf{Cpl}$ and $\mathcal{T}_{\mathcal{Z}} = \{\mathcal{Z} \cap J(X) : X \subseteq S\}$, then naturally the system $\mathcal{W}_{\mathcal{Z}} = < \mathcal{Z}, \mathcal{T}_{\mathcal{Z}} >$ is a subspace of $\mathcal{W}$.

LEMMA 2. (cf [1], Theorem 27). *Let* $T \in \mathbf{Th}$*. Then the lattice* $< L_T, \subseteq>$ *is isomorphic with the lattice* $< \mathcal{T}_{J(T)}, \subseteq>$*.*

PROOF. Let $T \in \mathbf{Th}$. Let us consider the mapping $H : L_T \longrightarrow \mathcal{T}_{J(T)}$ given by $H(Y) = J(Y)$, for every $Y \in L_T$. It is evident that H is an injection. We shall prove that it is a surjection. Let $\mathcal{Z} \in \mathcal{T}_{J(T)}$, i.e. $\mathcal{Z}$ is an open set in the space $\mathcal{W}_{J(T)}$. Since $J(T)$ is an open set in $\mathcal{W}$, then $\mathcal{Z}$ is also an open set in $\mathcal{W}$. From the definition of the space $\mathcal{W}$ there is a set $X \subseteq S$ such that $J(X) = \mathcal{Z}$. By Lemma 1 (a), (h) we conclude that $C(X) \in L_T$ and $H(C(X)) = \mathcal{Z}$. From Lemma 1 (h) it follows that H is an isomorphism. □

LEMMA 3. *$\mathcal{W}$ is homeomorphic with the Cantor space.*

PROOF. From Lemma 1 (b), (c) it follows that the set $\{J(\alpha) : a \in S\}$ is a basis of the space $\mathcal{W}$ and it consists of closed-open sets. Hence $\mathcal{W}$ is a zero-dimensional space. We easily prove that $\mathcal{W}$ is a regular space with a countable basis, so it is a metric space. Besides $\mathcal{W}$ is dense-in-itself and compact. It is well-known that every zero-dimensional metric, dense-in-itself, compact space is homeomorphic with the Cantor space. □

We note that the set $B_J = \{J(\alpha) : \alpha \in S\}$ is a Boolean algebra of all closed-open subsets of the Cantor space (see Lemma 7 (a)). Note also that B_J is isomorphic with the Lindenbaum-Tarski algebra $S/C(\emptyset)$.

Connections between Boolean algebras and the Cantor space are considered in [4].

LEMMA 4. *If* $T \in \mathbf{Th}\backslash\{C(\emptyset)\}$, *then the lattice* $< \mathcal{T}_{J(T)}, \subseteq>$ *is implicative and its content is equal to* INT.

PROOF. Let $T \in \mathbf{Th}\backslash\{C(\emptyset)\}$. Then $\mathcal{W}_{J(T)}$ is a non-empty, open subspace of $\mathcal{W}$. So $\mathcal{W}_{J(T)}$ is a dense-in-itself, metric space. The lattice $< \mathcal{T}_{J(T)}, \subseteq>$ of all open sets of $\mathcal{W}$ is implicative and, according to the well-known result of McKinsey and Tarski (see [3], chapter IX, Theorem 3.2), its content is equal to INT. □

PROOF OF THEOREM 1 follows from Lemmas 2 and 4.

Now we are going to prove Theorem 2.

LEMMA 5. *Any two non-empty, closed-open subsets of the Cantor space are homeomorphic. Any two open, but not closed subsets of the Cantor space are homeomorphic.* □

The next Lemma is an easy consequence of Lemmas 5 and 3:

LEMMA 6. *If* $\mathcal{Z}_1, \mathcal{Z}_2 \subseteq \mathbf{Cpl}$ *are both non-empty, closed-open sets in* $\mathcal{W}$ *or they are both open, but not closed sets in* $\mathcal{W}$ *then the lattices* $< \mathcal{T}_{\mathcal{Z}_1}, \subseteq>$ *and* $< \mathcal{T}_{\mathcal{Z}_2}, \subseteq>$ *are isomorphic.* □

LEMMA 7. *Let* $\mathcal{Z} \subseteq \mathbf{Cpl}$ *be an open set in* $\mathcal{W}$ *and let* $T \in \mathbf{Th}$ *be such that* $J(T) = \mathcal{Z}$. *Then*

(a) $\mathcal{Z}$ *is closed-open iff* $T \in \mathbf{Th}_0$

(b) $\mathcal{Z}$ *is open, but not closed iff* $T \in \mathbf{Th}_1$. □

PROOF OF THEOREM 2. Let $T_0 \in \mathbf{Th}_0\backslash\{C(\emptyset)\}$ and $T_1 \in \mathbf{Th}_1$. Take up an arbitrary $T \in \mathbf{Th}$. If $T \in \mathbf{Th}_0\backslash\{C(\emptyset)\}$, then $J(T)$ is a non-empty, closed-open set in $\mathcal{W}$ (see Lemma 1 (e) and Lemma 7). Hence by Lemmas 6 and 2 the lattice $< L_T, \subseteq>$ is isomorphic with the lattice $< L_{T_0}, \subseteq>$. We analogously prove that if $T \in \mathbf{Th}_1$, then $< L_T, \subseteq>$ is isomorphic with $< L_{T_1}, \subseteq>$. □

References

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